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On the Theory of Numbers and Generalized Quaternions.

L. E. Dickson.

§ 1. *Introduction and Summary.*

This paper presents new results and new methods concerning Diophantine equations, representation of integers by quaternary quadratic forms, and the theory of arithmetics of linear algebras. At the outset is given a simple new proof by the method of descent of Bachet's theorem that every positive integer is a sum of four integral squares. We next prove the generalization that the roots of the four squares may be chosen to satisfy four assigned linear congruences. The previous generalizations of Bachet's theorem relate to assigned values of the sum of the four roots.

Quaternions with integral coordinates are employed in § 3 to obtain a simpler proof of the writer's formulas giving all integral solutions of $x^2 + y^2 + z^2 + w^2 = uv$ and hence of $x^2 + y^2 + z^2 + w^2 + r^2 = s^2$.

An arithmetic of quaternions has been based on Lipschitz's definition of an integral quaternion as one having integral coördinates. Another was based on Hurwitz's definition of an integral quaternion as one whose four coördinates are either all integers or all halves of odd integers. The earlier investigations of these two arithmetics were based on the algorithm of division with a remainder whose norm is less than the norm of the divisor. We establish in § 4 the laws of factorization in these arithmetics without the aid of that algorithm, and in particular bring out clearly the essential difference between the integral quaternions of Lipschitz and those of Hurwitz.

The new method * is applied in § 5 to establish the laws of factorization in an arithmetic of generalized quaternions for which no previous method had been suggested since it is impossible to find the (existing) greatest common left divisor by an algorithm of divisions with remainders of decreasing norms. The laws of divisibility are entirely similar to those of Hurwitz's integral quaternions. It follows that every positive integer can be represented in the form

$$N = x^2 + xz + 2z^2 + y^2 + yw + 2w^2,$$

* In a paper offered to the Bulletin of the American Mathematical Society, this method has been applied to the simpler problem (hitherto treated only by ideals) of proving the uniqueness of factorization of integral algebraic numbers of any quadratic field of class unity.

while all integral solutions of $N = uv$ are obtained explicitly. Hence every multiple of 4 can be represented in the form $\xi^2 + \eta^2 + 7z^2 + 7w^2$ with $\xi + z$ even.

It will be shown on another occasion that the present new methods are applicable to various other algebras.

§ 2. *Proof by descent that every positive integer is a sum of four integral squares, with generalizations.*

THEOREM 1. *If p is a given positive integer and α and β are given integers such that $\alpha^2 + \beta^2 + 1$ is divisible by p, we can find integers Z, W, U, V, such that*

$$(1) \qquad p = Z^2 + W^2 + (pU + \alpha W - \beta Z)^2 + (pV + \beta W + \alpha Z)^2.$$

We may write $\alpha = Ap + a$, $\beta = Bp + b$, where

$$(2) \qquad |a| \leqq \tfrac{1}{2}p, \quad |b| \leqq \tfrac{1}{2}p.$$

Then (1) is equivalent to

$$(3) \qquad p = Z^2 + W^2 + (pz + aW - bZ)^2 + (pw + bW + aZ)^2,$$

where $z = U + AW - BZ$, $w = V + BW + AZ$. By hypothesis,

$$(4) \qquad a^2 + b^2 + 1 = pq,$$

where q is an integer. By (2), $q < p$ unless $p = 1$. We shall prove that (3) follows from the similar formula

$$(5) \qquad q = z^2 + w^2 + (qZ + aw - bz)^2 + (qW + bw + az)^2,$$

which differs from (3) only by the interchange of p with q, z with Z, and w with W. The expansion of (5) gives

$$q = (a^2 + b^2 + 1)(z^2 + w^2) + q^2(Z^2 + W^2) + 2qZ(aw - bz) + 2qW(bw + az).$$

By use of (4) and multiplication by p/q, we get

$$p = p^2(z^2 + w^2) + pq(Z^2 + W^2) + 2pz(aW - bZ) + 2pw(bW + aZ).$$

Completing the square of the terms in z and those in w, and noting that the new coefficient $pq - a^2 - b^2$ of $Z^2 + W^2$ is unity by (4), we get (3). Since (1) therefore follows from (5) in which $a^2 + b^2 + 1$ is divisible by q, while $q < p$, our theorem follows by descent. In fact, it is true when $p = 1$ since then $a = b = 0$ by (2), while (3) then holds for $Z = 1$, $W = z = w = 0$.

LEMMA 1. *If p is an odd prime not dividing B,*

$$x^2 - By^2 - C \equiv 0 \pmod{p} \tag{6}$$

has integral solutions x, y.

For $x = 0, 1, \cdots, \frac{1}{2}(p-1)$, x^2 takes $\frac{1}{2}(p+1)$ incongruent values modulo p since neither the sum nor difference of two values of x is divisible by p. Hence for $y = 0, 1, \cdots, \frac{1}{2}(p-1)$, y^2 and therefore also $By^2 + C$ takes $\frac{1}{2}(p+1)$ incongruent values modulo p. At least one of the values of x^2 is congruent to one of the values of $By^2 + C$ since otherwise there would be $2 \cdot \frac{1}{2}(p+1)$, or more than p, incongruent numbers modulo p.

In particular, $x^2 + y^2 + 1 \equiv 0 \pmod{p}$ has solutions. Hence by Theorem 1, every prime is a sum of four integral squares. The product of two such sums is another such sum.* Hence *every positive integer is the sum of four integral squares.*

LEMMA 2. *If m is odd or double an odd integer, and if B and C are each prime to m, there exist solutions of*

$$x^2 - By^2 - C \equiv 0 \pmod{m}. \tag{7}$$

This will follow for $m = MN$ if true for the relatively prime moduli M and N. For, let

$$x_0^2 - By_0^2 - C \equiv 0 \pmod{M}, \qquad x_1^2 - By_1^2 - C \equiv 0 \pmod{N}.$$

We can determine integers x, y such that

$$x \equiv x_0,\ y \equiv y_0 \pmod{M}, \qquad x \equiv x_1,\ y \equiv y_1 \pmod{N}.$$

Then $x^2 - By^2 - C$ is divisible by M and N and hence by m. Thus Lemma 2 will follow from Lemma 1 if we prove that when $x_2^2 - By_2^2 - C = p^k q$ for $k \geqq 1$, $p > 2$, (7) has solutions $x = x_2 + p^k\xi$, $y = y_2 + p^k\eta$ modulo p^{k+1}. The condition is $q + 2x_2\xi - 2By_2\eta \equiv 0 \pmod{p}$ and is satisfied by choice of integers ξ, η (one zero) since x_2 and y_2 are not both divisible by p (C being prime to p).

In particular, $\alpha^2 + \beta^2 + 1 \equiv 0 \pmod{p}$ has solutions if p is odd or double an odd integer (but not if p is divisible by 4). Hence by Theorem 1 any such p is a sum of four squares (1), the roots of two † of which are con-

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* The product of the norms of two quaternions is the norm of their product.

† Say X, Y. Is *every* representation of p obtained from (1)? In case $Z^2 + W^2$ is prime to p, the conditions $\alpha W - \beta Z \equiv X$, $\beta W + \alpha Z \equiv Y \pmod{p}$ determine α and β, and their values satisfy $\alpha^2 + \beta^2 + 1 \equiv 0 \pmod{p}$. For,

$$(\alpha^2 + \beta^2)(W^2 + Z^2) \equiv X^2 + Y^2 = p - (W^2 + Z^2),$$
$$(\alpha^2 + \beta^2 + 1)(W^2 + Z^2) \equiv 0 \pmod{p}.$$

gruent modulo p to assigned linear functions $\alpha W - \beta Z$ and $\beta W + \alpha Z$ of the remaining roots Z and W. This result is the special case $a = 1$, $b = 0$, $c = \beta$, $d = -\alpha$ of

THEOREM 2. *If p is a positive integer and if a, b, c, d are any integers the sum of whose squares is divisible by p, there exist integers x, y, z, w such that*

$$p = x^2 + y^2 + z^2 + w^2, \tag{8}$$

$$\left.\begin{aligned} A &= ax - by - cz - dw \equiv 0, \quad B = bx + ay + dz - cw \equiv 0, \\ C &= cx - dy + az + bw \equiv 0, \quad D = dx + cy - bz + aw \equiv 0 \end{aligned}\right\} \pmod{p}. \tag{9}$$

In other words, *if the norm of the quaternion $\eta = a + bi + cj + dk$ is divisible by p, there exist quaternions ξ and π, also having integral coördinates, such that* $p = \pi\pi'$, $\eta = \pi\xi$, where π' is the conjugate of π.* For, if we write $\pi' = x + yi + zj + wk$, we see that the coördinates of $\pi'\eta$ are A, B, C, D in (9). Hence $\pi'\eta = p\xi = \pi'\pi\xi$, whence $\eta = \pi\xi$.

The theorem is true for $p = p_1p_2$ if true for p_1 and p_2. For, since $N(\eta)$ is divisible by p and hence by p_1, we have $\eta = \pi_1\xi$, $p_1 = \pi_1\pi_1'$. Then $N(\xi) = N(\eta)/p_1$ is divisible by p_2, so that $\xi = \pi_2\zeta$, $p_2 = \pi_2\pi_2'$. Thus $\eta = (\pi_1\pi_2)\zeta$, $N(\pi_1\pi_2) = p_1p_2 = p$. Hence the theorem will be true for every positive integer p if proved true for every prime p.

In the proof of Theorem 2 when p is a prime, we may assume that a, b, c, d are not all divisible by p (since otherwise the theorem states merely that p is a sum of four squares), and that $a^2 + b^2$ is not divisible by p. For if any one of $a^2 + b^2$, $a^2 + c^2$, $a^2 + d^2$ be not divisible by p, we make use of the substitution $(bcd)(yzw)(BCD)$ which permutes b, c, d cyclically, etc., and leaves the system (8) and (9) unaltered. But if $a^2 + b^2 \equiv a^2 + c^2 \equiv a^2 + d^2 \equiv 0 \pmod{p}$, $b^2 \equiv c^2 \equiv d^2 \equiv -a^2$. But $a^2 + \cdots + d^2$ is divisible by p. Hence $2a^2 \equiv 0$. If $p > 2$, a, b, c, d are all divisible by p, contrary to hypothesis. If $p = 2$, the same conclusion holds unless a, b, c, d are all odd and then (8) and (9) hold with $x = y = 1$, $z = w = 0$.

The pair of congruences $A \equiv 0$, $B \equiv 0$ is equivalent to the pair

$$(a^2 + b^2)x \equiv ez + fw, \quad (a^2 + b^2)y \equiv -fz + ew, \tag{10}$$
$$e = ac - bd, \quad f = ad + bc.$$

From (10) we get

In particular, (1) gives all representations of a prime p except those in which $W^2 + Z^2 = p$ or 0 (arising only if $p = 4n + 1$).

* In a companion theorem, we have $p = \pi\pi'$, $\eta = \xi\pi$.

$$(a^2+b^2)C \equiv (a^2+b^2+c^2+d^2)(az+bw),$$
$$(a^2+b^2)D \equiv (a^2+b^2+c^2+d^2)(aw-bz),$$

which imply $C \equiv 0$, $D \equiv 0 \pmod{p}$. Since a^2+b^2 is prime to p, there exist integers r and s such that

$$1 = r(a^2+b^2) - sp. \tag{11}$$

Multiplication of (10) by r gives

$$x = r(ez+fw) + pV, \quad y = r(-fz+ew) + pU, \tag{12}$$

where V and U are integers. Write $\alpha = re$, $\beta = rf$, $Z = z$, $W = w$. Then (8) becomes (1), while

$$\alpha^2+\beta^2+1 = 1 + r^2(e^2+f^2) = 1 + r^2(a^2+b^2)(c^2+d^2)$$
$$\equiv 1 - r^2(a^2+b^2)^2 \equiv 0 \pmod{p}.$$

Since (1) has integral solutions Z, W, U, V, the integers $z = Z$, $w = W$, and x, y in (12) satisfy equation (8) and congruences (9).

§ 3. *Quaternions and Diophantine Equations.*

LEMMA 3. *Let p be a rational prime which divides AB and $N(A)$, but not A, where A and B are quaternions with integral coördinates, those of A being not all odd if $p = 2$. By Theorem 2, there exist quaternions P and Q with integral coördinates such that $A = QP$, $PP' = p$. Then PB is divisible by p.*

Write

$$P = l + mi + nj + qk, \quad Q = r + si + tj + uk, \quad B = \alpha + \beta i + \gamma j + \delta k.$$

Then $A = QP = a + bi + cj + dk$, where

$$\begin{aligned} a &= rl - sm - tn - uq, & b &= rm + sl + tq - un, \\ c &= rn - sq + tl + um, & d &= rq + sn - tm + ul. \end{aligned} \tag{13}$$

Permuting i, j, k cyclically if necessary, we may assume as in the proof of Theorem 2 that a^2+b^2 is not divisible by p. Since $AB = E + Fi + \cdots$ is divisible by p,

$$E = a\alpha - b\beta - c\gamma - d\delta \equiv 0,$$
$$F = b\alpha + a\beta + c\delta - d\gamma \equiv 0 \pmod{p}. \tag{14}$$

From these we get

$$(a^2+b^2)\alpha \equiv \rho\gamma + \sigma\delta, \quad (a^2+b^2)\beta \equiv \sigma\gamma - \rho\delta \pmod{p}, \tag{15}$$

where $\rho = ac + bd$, $\sigma = ad - bc$. We are to prove that the coördinates of

$PB = \lambda + \mu i + \tau j + \omega k$ are divisible by p:

$$\lambda = l\alpha - m\beta - n\gamma - q\delta, \quad \mu = l\beta + m\alpha + n\delta - q\gamma,$$
$$\tau = l\gamma - m\delta + n\alpha + q\beta, \quad \omega = l\delta + m\gamma - n\beta + q\alpha.$$

First, let $l^2 + m^2$ be not divisible by p. Then the pair of congruences $\lambda \equiv 0$, $\mu \equiv 0$ is equivalent to the pair

$$(16) \quad (l^2 + m^2)\alpha \equiv U\gamma + V\delta, \quad (l^2 + m^2)\beta \equiv V\gamma - U\delta,$$

where $U = ln + mq$, $V = lq - mn$. As below (10), $\tau \equiv 0$, $\omega \equiv 0$ follow from (16). Multiply (16) by $a^2 + b^2$ and eliminate α and β by means of (15). Hence we are to prove that

$$\gamma X + \delta Y \equiv 0, \quad \gamma Y - \delta X \equiv 0 \pmod{p},$$

where

$$X = (l^2 + m^2)\rho - (a^2 + b^2)U, \quad Y = (l^2 + m^2)\sigma - (a^2 + b^2)V.$$

By (13),

$$\rho = (r^2 + s^2 - t^2 - u^2)U + 2(ru - st)(ml - nq) + (rt + su)(l^2 + q^2 - m^2 - n^2),$$
$$a^2 + b^2 = (r^2 + s^2)(l^2 + m^2) + 2(rt + su)(mq - ln) + 2(st - ru)(mn + lq) + (t^2 + u^2)(n^2 + q^2).$$

In the final terms, replace $l^2 + q^2$ by $-(m^2 + n^2)$ and $n^2 + q^2$ by $-(l^2 + m^2)$, which is permissible since $N(P) = p \equiv 0$. Hence

$$X \equiv 2(ru - st)[(l^2 + m^2)(ml - nq) + (mn + lq)U] - 2(rt + su)[(l^2 + m^2)(m^2 + n^2) + (mq - ln)U].$$

The quantities in brackets are equal to

$$ml(l^2 + m^2 + n^2 + q^2), \quad m^2(l^2 + m^2 + n^2 + q^2),$$

respectively and hence are divisible by $N(P) = p$. Next,

$$\sigma = (r^2 + s^2 - t^2 - u^2)V - 2(rt + su)(ml + nq) + (ru - st)(l^2 + n^2 - m^2 - q^2).$$

Replace $l^2 + n^2$ by $-m^2 - q^2$. We see that

$$Y = -2(ru - st)[(l^2 + m^2)(m^2 + q^2) - (mn + lq)V] - 2(rt + su)[(l^2 + m^2)(ml + nq) + (mq - ln)V].$$

The quantities in brackets are equal to $m^2N(P)$ and $mlN(P)$.

Second, let $l^2 + m^2 \equiv 0 \pmod{p}$. Since $p = l^2 + m^2 + n^2 + q^2$, either $l^2 + m^2 = p$, $n = q = 0$, or $l^2 + m^2 = 0$, $l = m = 0$, $n^2 + q^2 \equiv 0 \pmod{p}$.

In either case the above expression for a^2+b^2 is divisible by p, contrary to hypothesis.

LEMMA 4. *In the excluded case $p=2$, a, b, c, d all odd, Lemma 3 holds for a properly chosen P, but not for all P's.*

By the first congruence (14), $\alpha+\beta+\gamma+\delta$ is even. Hence at least one of the sums $\alpha+\beta$, $\alpha+\gamma$, $\alpha+\delta$ is even since their sum is even. Permuting i, j, k cyclically, we may take $\alpha+\beta$ even. Then

$$B=(1-i)R, \quad R=\tfrac{1}{2}(a-\beta)+\tfrac{1}{2}(a+\beta)i+\tfrac{1}{2}(\gamma-\delta)j+\tfrac{1}{2}(\gamma+\delta)k,$$
$$A=Q(1+i), \quad Q=\tfrac{1}{2}(a+b)+\tfrac{1}{2}(b-a)i+\tfrac{1}{2}(c-d)j+\tfrac{1}{2}(c+d)k,$$

where R and Q have integral coördinates. Hence $1+i$ is a choice of P such that $A=QP$, $PP'=2$, $PB=2R$.

For $A=1+i+j-k$, $B=j-k$, $AB=2(j-1)$, $N(A)=4$. Here $\alpha=\beta=0$ and as above we may choose $P=1+i$, whence $Q=1+j$, $R=j$. But the choice $1-k$ of P is excluded although $A=(1+i)P$, since now $PB=-1+i+j-k$ is not divisible by 2.

THEOREM 3. *If a rational prime p divides AB, $N(A)$ and $N(B)$, where A and B are quaternions with integral coördinates, there exist quaternions P, Q, R with integral coördinates such that $A=QP$, $B=P'R$, $PP'=p$.*

This is true by Lemmas 3 and 4 if A is not divisible by p. Next, let $A=Tp$. Applying Theorem 2 to $N(B)\equiv 0 \pmod p$, we obtain quaternions P' and R with integral coördinates such that $B=P'R$, $P'P=p$. Write Q for TP'. Then $A=T\cdot P'P=QP$, and $PP'=p$.

THEOREM 4. *All integral solutions of $x^2+y^2+z^2+w^2=uv$ are given by*

$$(17) \qquad x+yi+zj+wk=fAB, \quad u=fN(A), \quad v=fN(B),$$

in which f is an arbitrary integer, while the eight coördinates of quaternions A and B are arbitrary integers without a common factor >1.

First, all rational solutions are given by (17) in which f is now rational, and we may take B to be an integer α. If $v=0$, take $B=0$. If $v\neq 0$, we may write

$$\frac{x}{v}=\frac{a}{\alpha}, \quad \frac{y}{v}=\frac{b}{\alpha}, \quad \frac{z}{v}=\frac{c}{\alpha}, \quad \frac{w}{v}=\frac{d}{\alpha}, \quad f=\frac{v}{\alpha^2},$$

where a, b, c, d, α are integers without a common factor >1.

Writing $A=a+bi+cj+dk$, we have

$$x + yi + zj + dk = \frac{v}{a} A = faA = fAB, \quad v = fN(B),$$

whence $uv = N(x + yi + \cdots) = f^2N(A)N(B), \quad u = fN(A)$.

Second, to prove that our formulas give all integral solutions when we restrict f to integral values, it suffices to consider any prime factor of the denominator of the irreducible fraction f and to prove that the quotients of AB, $N(A)$, $N(B)$ by p may be expressed in the form QR, $N(Q)$, $N(R)$, where Q and R are quaternions with integral coördinates. The latter is true by Theorem 3 since $AB = QPP'R = pQR$.

§ 4. *Theory of integral quaternions.*

INTEGRAL QUATERNIONS. We seek a definition of integral quaternions and hence of the term "divisible by," such that we shall have

LEMMA 5. *If Q, P, B are any integral quaternions and p is any rational prime such that $PP' = p$ and QPB is divisible by p, either QP or PB is divisible by p.*

If we define an integral quaternion to be one whose four coördinates are all integers, we see that Lemma 3 implies Lemma 5 except for the case in which $p = 2$ and the coördinates of QP are all odd integers. But in that case, Lemma 5 is not always true. For, by the final example under Lemma 4, $P = 1 - k$, $Q = 1 + i$, $B = j - k$, $PP' = p = 2$, we have $QPB = 2(j - 1)$, and neither $QP = 1 + i + j - k$ nor $PB = -1 + i + j - k$ is divisible by 2. But both become divisible by 2, with integral quaternion quotients, if we define an integral quaternion to be one whose four coördinates are either all integers or all halves of odd integers. Under this definition due to Hurwitz, Lemma 5 holds in all cases. The proof is as follows.

If a is such an integral quaternion, we can find * units u and v such that au and va have integral coördinates. Hence we can find units u, v, w such that $Q = uQ_1$, $P = P_1v$, $vB = B_1w$, where Q_1, P_1 and B_1 have integral coördinates. Since $QPB = uQ_1P_1B_1w$ is divisible by p, $Q_1P_1B_1$ is divisible by p. First, let $p > 2$. Since we know that Lemma 5 holds for quaternions with integral coördinates, we conclude that either $Q_1P_1 = u'QPv'$ or $P_1B_1 = PBw'$ is divisible by p, whence either QP or PB is divisible by p.

Finally, let $p = 2$, $PP' = 2$. Any integral quaternion whose norm is even has † integral coördinates and can be expressed in each of the forms $(1 + i)h$, $l(1 + i)$, where h and l are integral quaternions. Hence $P =$

* *Algebras and Their Arithmetics*, p. 153.

† *Ibid.*, p. 154.

$(1+i)u = v(1+i)$, where u and v are units. If $N(Q)$ is even, $Q = q(1+i)$ and QP is divisible by 2. If $N(Q)$ is odd, $N(QPB)$ is divisible by 4 by hypothesis, so that $N(B)$ is even and $B = (1+i)R$, whence PB is divisible by 2.

LEMMA 6. *Any Hurwitz integral quaternion A whose norm is divisible by the rational prime p has in common with p a greatest common left divisor D which is unique apart from a unit right factor.*

If A is divisible by p, evidently $D = p$. Next, let A be not divisible by p. Choose a unit u such that uA has integral coördinates. By Theorem 2, $uA = P_1S$, $P_1P'_1 = p$. Write $P = u'P_1$. Then $A = PS$, $PP' = p$. Suppose that also $A = \pi R$, $\pi\pi' = p$, whence $\pi'\pi = p$. Then $\pi'PS = pR$ is divisible by p. Since $PS = A$ is not divisible by p, Lemma 5 shows that $\pi'P$ is the product of p by an integral quaternion T. Then $\pi'P = \pi'\pi T$ and $P = \pi T$. By their norms, $p = pN(T)$, so that T is a unit. Finally, if $p = \lambda\mu$, where λ and μ are integral quaternions neither a unit, evidently $p = N(\lambda) = N(\mu)$, whence λ and μ are indecomposable. Hence every left divisor of p is the product of p itself or P by a unit, where P is one of the prime integral quaternions for which $PP' = p$.

THEOREM * 5. *Any two Hurwitz integral quaternions a and b have a greatest common left divisor (a, b) which is unique up to a unit right factor.*

We may write $a = da u$, $b = d\beta v$, where d is an integer, u and v are units, and the eight integral coördinates of α and β have no common integral divisor > 1. If $N(\alpha)$ and $N(\beta)$ are relatively prime, $(a, b) = d$. If they have the common prime divisor p, we may assume that α, for example, is not divisible by p. Then Lemma 6 shows that α has in common with p the greatest left prime quaternion divisor P with integral coördinates which is unique up to a unit right factor. If P is also a left divisor of β, we call p a favorable prime and proceed with the integral quaternions α/P and β/P as with the initial α and β. In the contrary case, we discard the unfavorable p and start with a new prime common divisor of $N(\alpha)$ and $N(\beta)$. To fix the ideas, let this process give $\alpha = PQRA$, $\beta = PQRB$, where $N(A)$ and $N(B)$ have no favorable common prime factor, so that A and B have no common left divisor not a unit.

Any common integral divisor > 1 of the coördinates of PQR would divide both α and β, contrary to hypothesis. If the primes $p = N(P)$, $q = N(Q)$,

* If we define integral quaternions to be those having integral coördinates, we see from the proof of Lemma 5 that Theorem 5 now holds provided at least one of a and b is of odd norm. This was first proved otherwise by the writer in *Proc. London Math. Soc.*, (2), Vol. 20 (1921), pp. 225-32.

$r = N(R)$ are arranged in any new order, as q, r, p, there exist * prime quaternions Q_1, R_1, P_1 of norms q, r, p and unique up to unit right factors such that $Q_1R_1P_1 = PQR$. Thus α and β have the common left divisor $Q_1R_1P_1$. Hence our initial process on α and β may be performed with q, r, p as successive favorable primes and again terminates after three steps. Hence (a, b) is $dPQR$ or its product by a unit.

LEMMA 7. *If a rational prime p divides AB, $N(A)$ and $N(B)$, where A and B are integral quaternions, there exist integral quaternions P, Q, R such that $A = QP$, $B = P'R$, $PP' = p$.*

As in the proof of Lemma 6, $A = QP$, $PP' = p$. If A is not divisible by p, Lemma 5 shows that PB is the product of p by an integral quaternion R, whence $B = P'R$. If A is divisible by p, the same result follows as in the proof of Theorem 3.

Every integral quaternion is of the form

$$q = x\rho + yi + zj + wk, \qquad \rho = \tfrac{1}{2}(1 + i + j + k),$$

where x, y, z, w are integers. The norm of q is the left member of (18). As in the proof of Theorem 4, Lemma 7 implies

THEOREM 6. *All integral solutions of*

$$x^2 + xy + xz + xw + y^2 + z^2 + w^2 = uv \tag{18}$$

are given by

$$x\rho + yi + zj + wk = fAB, \quad u = fN(A), \quad v = fN(B),$$

where f is an arbitrary integer, while the eight coefficients of ρ, i, j, k in A and B are integers without a common factor > 1.

§ 5. *An arithmetic of generalized quaternions.*

As our integral elements we take $X = x + yi + zH + wL$, where x, y, z, w are integers and the units i, H, L have the multiplication table †

$$\begin{gathered} i^2 = -1, \; iH = L, \; iL = -H, \; Hi = i - L, \; Li = H - 1, \\ H^2 = H - \epsilon, \; HL = \epsilon i, \; LH = L - \epsilon i, \; L^2 = -\epsilon, \end{gathered} \tag{19}$$

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* *Algebras and Their Arithmetics*, p. 155. The proof requires only our present Lemma 6.

† *Algebras and Their Arithmetics*, pp. 187-194. We take $\beta \equiv 3 \pmod 4$ and write $\epsilon = (1 + \beta)/4$, $H = H_4$, $L = H_3$, choosing as integral elements those of set S_2, p. 191.

where ϵ is an integer. Define the conjugate of X to be $X' = 2x + z - X$, and the norm of X to be

$$N(X) = XX' = x^2 + xz + y^2 + yw + \epsilon z^2 + \epsilon w^2. \tag{20}$$

The norm of a product is the product of the norms.

THEOREM 7. *If p is a prime and if $\eta = a + bi + cH + dL$ is any integral element not divisible by p, whose norm is divisible by p,*

$$N(\eta) = a^2 + ac + b^2 + bd + \epsilon c^2 + \epsilon d^2 \equiv 0 \pmod{p}, \tag{21}$$

and if $\epsilon = 2$, there exists an integral element $\pi' = x + yi + zH + wL$ of norm p such that $\eta\pi'$ is divisible by p, whence $\eta = \xi\pi$, ξ being an integral element.

By (19), $\eta\pi' = A + Bi + CH + DL$, where

$$\begin{aligned} A &= ax - by - dy - \epsilon cz - \epsilon dw, & B &= bx + ay + cy - \epsilon dz + \epsilon cw, \\ C &= cx + dy + az + cz - bw, & D &= dx - cy + bz + dz + aw. \end{aligned} \tag{22}$$

We are to prove the existence of integers x, y, z, w such that A, B, C, D are all divisible by p and such that the norm (20) is equal to p. The proof will be reduced to the case $a = 1$, $d = 0$. Then (21) becomes

$$1 + c + b^2 + \epsilon c^2 \equiv 0 \pmod{p}, \tag{23}$$

while $A \equiv 0$, $D \equiv 0 \pmod{p}$ give $x \equiv by + \epsilon cz$, $w \equiv cy - bz$, and then $B \equiv 0$, $C \equiv 0 \pmod{p}$ are satisfied in view of (23). Hence we desire to prove the existence of solutions of the form

$$y = Z,\ z = Y,\ x = pz + bZ + \epsilon cY,\ w = py + cZ - bY \tag{24}$$

of $p = N(\pi)$. Completing the square of the terms in x in $4N(\pi)$, we see that we are to prove (for the case of primes p)

LEMMA 8. *If p is any positive integer and if c and b are any integral solutions of congruence (23), there exist integers Z, Y, z, y such that, for $\epsilon = 2$,*

$$\begin{aligned} 4p = [2pz + 2bZ + (2\epsilon c + 1)Y]^2 + 4Z^2 + 4Z(py + cZ - bY) \\ + (4\epsilon - 1)Y^2 + 4\epsilon(py + cZ - bY)^2. \end{aligned} \tag{25}$$

Expanding (25) and employing

$$1 + c + \epsilon c^2 + b^2 = pq, \tag{26}$$

we get

$$(27)\quad 4p = 4p^2z^2 + 4\epsilon p^2y^2 + 8pbzZ + 4p(2\epsilon c + 1)(zY + yZ) - 8p\epsilon byY + 4pqZ^2 + 4\epsilon pqY^2.$$

The addition of multiples of p to b and c merely replaces z and y in (24) by new integers. If $p > 3$, we may take $|c| \leqq \frac{1}{2}p$, $|b| \leqq \frac{1}{2}p$; then

$$N \equiv 1 + c + 2c^2 + b^2 \leqq 1 + \tfrac{1}{2}p + \tfrac{3}{4}p^2 < p^2, \text{ since } 5 < (p-1)^2.$$

If $p = 3$, we may take $|c| \leqq 1$, $|b| \leqq 1$, whence $N \leqq 5 < 3^2$. If $p = 2$, either $c = 0$, $N \leqq 2 < 2^2$, or we take $c = -1$, whence $N \leqq 3 < 2^2$. Hence if $p > 1$, $N < p^2$, $q < p$. We shall prove (27) by descent.

We prove that (27) follows from the similar formula

$$(28)\quad 4q = 4q^2Z^2 + 4\epsilon q^2Y^2 + 8qbZz + 4q(2\epsilon c + 1)(Zy + Yz) - 8q\epsilon bYy + 4qpz^2 + 4\epsilon qpy^2,$$

which is obtained from (27) by interchanging p and q, y and Y, z and Z. Multiply (28) by p/q; we obtain (27). Finally, if $p = 1$, we may take $b = c = 0$, and see that (27) holds when $y = Y = Z = 0$, $z = 1$. Hence the Lemma is proved.

In the proof of Theorem 7, we consider three cases.

(i) First, let $a^2 + \epsilon d^2$ be not divisible by the prime p. Then $A \equiv 0$, $D \equiv 0$ are equivalent to

$$(29)\quad (a^2 + \epsilon d^2)x \equiv ey + \epsilon fz, \quad (a^2 + \epsilon d^2)w \equiv fy - ez \pmod p,$$

where $e = a(b + d) + \epsilon cd$, $f = ac - d(b + d)$. Then

$$(a^2 + \epsilon d^2)B = (ay - \epsilon dz)N(\eta), \quad (a^2 + \epsilon d^2)C = (az + dy)N(\eta),$$

so that $B \equiv 0$, $C \equiv 0 \pmod p$. Determine integers r, s so that

$$(30)\qquad 1 = r(a^2 + \epsilon d^2) - sp.$$

Write $\beta = re$, $\gamma = rf$, $y = Z$, $z = Y$, and multiply (29) by r. Thus

$$x = pz + \beta Z + \epsilon yY, \quad w = py + yZ - \beta Y,$$

which are (24) with β, γ in place of b, c. By Lemma 8, $p = N(\pi)$ if

$$(31)\qquad 1 + \gamma + \epsilon\gamma^2 + \beta^2 \equiv 0 \pmod p.$$

We have $\quad e^2 + \epsilon f^2 = (a^2 + \epsilon d^2)[(b + d)^2 + \epsilon c^2]$.

Multiply by r^2, apply (30), and insert in (31), viz.,

$$1 + rf + r^2(e^2 + \epsilon f^2) \equiv 1 + r[f + (b + d)^2 + \epsilon c^2] \pmod p.$$

The quantity in brackets is equal to $N(\eta) - a^2 - \epsilon d^2$, whose product by r is congruent to -1 by (30).

(ii) Second, let $b^2 + \epsilon c^2$ be not divisible by p. This case is reduced * to the first by interchanging a with b and c with d, and changing the signs of y and w, whence A and B are interchanged, and likewise C and D, while (21) is unaltered.

(iii) Let $a^2 + \epsilon d^2 \equiv 0$, $b^2 + \epsilon c^2 \equiv 0$, whence $ac + bd \equiv 0$ by (21). Now $\eta\pi' = \zeta P$, where $P = -i\pi'$ is of norm p and $\zeta = \eta i = -b - d + (a+c)i + dH - cL$. Since the latter may be taken in place of η, we may replace a, b, c, d by $-b-d, a+c, d, -c$, respectively. Our three congruences become

$$(b+d)^2 + \epsilon c^2 \equiv 0, \quad (a+c)^2 + \epsilon d^2 \equiv 0, \quad d(b+d) + c(a+c) \equiv 0.$$

Subtracting the initial second, first and third congruences respectively, we get

$$2bd + d^2 \equiv 0, \quad 2ac + c^2 \equiv 0, \quad d^2 + c^2 \equiv 0.$$

If $c \equiv 0$, then $b \equiv 0$, $d \equiv 0$, $a \equiv 0$, contrary to hypothesis. Hence neither c nor d is divisible by p, whence $d \equiv -2b$, $c \equiv -2a$, $p \neq 2$. Thus $a^2 + b^2 \equiv 0$, $a \not\equiv 0$. Then

$$0 \equiv a^2 + \epsilon d^2 \equiv a^2 + 4\epsilon b^2 \equiv a^2(1 - 4\epsilon), \quad 4\epsilon \equiv 1 \pmod{p}.$$

Since $\epsilon = 2$, we have $p = 7$, and -1 is a quadratic non-residue of 7. This contradicts $a^2 + b^2 \equiv 0$, $a \not\equiv 0$. Hence case (iii) is excluded.

LEMMA 9. *For any prime p, congruence* (23) *is solvable if* $\epsilon = 2$.

For $p = 2$, take $c \equiv -1 - b^2$. For $p > 2$, we multiply (23) by 8 and get

$$(4c+1)^2 + 8b^2 + 7 \equiv 0 \pmod{p}.$$

This has solutions by Lemma 1.

By Lemmas 8 and 9, every prime is the norm of an integral element. Hence every positive integer is a norm.

THEOREM 8. *For $\epsilon = 2$ every positive integer is expressible as the norm* (20) *of an integral element.*

Since the quadruple of (20) with $\epsilon = 2$ is

$$\xi^2 + \eta^2 + 7z^2 + 7w^2, \quad \xi = 2x + z, \quad \eta = 2y + w,$$

it follows † that *every multiple of* 4 *is expressible in the form* $\xi^2 + \eta^2 + 7z^2 + 7w^2$, with $\xi \equiv z$, $\eta \equiv w \pmod 2$.

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* Also by the replacement used in the proof of Theorem 9 which replaces A, B, C, D by $B, -A, D, -C$.

† Cf. Klein-Fricke, Elliptische Modulfunctionen, II, 1892, p. 400; G. Humbert, Comptes Rendus, Paris, 169, 1919, pp. 407-14.

THEOREM 9. *Let p be a rational prime which divides AB and $N(A)$, but not A, where A and B are integral elements. For $\epsilon = 2$, there exist by Theorem 7 integral elements P and Q such that $A = QP$, $PP' = p$. Then PB is divisible by p.*

Write $A = a + bi + cH + dL$. The theorem is true if true when A is replaced by $-iA$, which is derived from A by replacing a, b, c, d by $b, -a, d, -c$, respectively. Then $a^2 + \epsilon d^2$ and $b^2 + \epsilon c^2$ are interchanged. If both are divisible by p, we use Ai instead of A as in case (iii) above, and conclude that we may assume that $a^2 + \epsilon d^2$ is prime to p. Write

$$P = l + mi + nH + qL, \quad Q = r + si + tH + uL,$$
$$B = \alpha + \beta i + \gamma H + \delta L.$$

From $A = QP$, and $AB = E + Fi + GH + KL$, we get

$$(32) \quad \begin{aligned} a &= rl - sm - um - \epsilon tn - \epsilon uq, & c &= tl + um + rn + tn - sq, \\ b &= sl + rm + tm - \epsilon un + \epsilon tq, & d &= ul - tm + sn + un + rq; \end{aligned}$$

$$(33) \quad \begin{aligned} E &= a\alpha - b\beta - d\beta - \epsilon c\gamma - \epsilon d\delta, & G &= c\alpha + d\beta + a\gamma + c\gamma - d\delta, \\ F &= b\alpha + a\beta + c\beta - \epsilon d\gamma + \epsilon c\delta, & K &= d\alpha - c\beta + b\gamma + d\gamma + a\delta. \end{aligned}$$

The expressions (33) are divisible by p by hypothesis. From $E \equiv K \equiv 0$,

$$(34) \quad (a^2 + \epsilon d^2)\alpha \equiv \rho\beta + \epsilon\sigma\gamma, \quad (a^2 + \epsilon d^2)\delta \equiv \sigma\beta - \rho\gamma \pmod{p},$$

where $\rho = a(b + d) + \epsilon cd$, $\sigma = ac - d(b + d)$. We are to prove that p divides the four coefficients of $PB = R + Si + TH + WL$,

$$(35) \quad \begin{aligned} R &= l\alpha - m\beta - q\beta - \epsilon n\gamma - \epsilon q\delta, & T &= n\alpha + q\beta + l\gamma + n\gamma - m\delta, \\ S &= m\alpha + l\beta + n\beta - \epsilon q\gamma + \epsilon n\delta, & W &= q\alpha - n\beta + m\gamma + q\gamma + l\delta. \end{aligned}$$

(i) Let $l^2 + \epsilon q^2$ be not divisible by p. Then $R \equiv 0$, $W \equiv 0$ are equivalent to

$$(36) \quad \begin{gathered} (l^2 + \epsilon q^2)\alpha \equiv U\beta + \epsilon V\gamma, \quad (l^2 + \epsilon q^2)\delta \equiv V\beta - U\gamma \pmod{p}, \\ U = l(m + q) + \epsilon nq, \quad V = ln - q(m + q). \end{gathered}$$

Now (36) imply $S \equiv 0$, $T \equiv 0$ for the same reason that (29) implied $B \equiv 0$, $C \equiv 0$, since the replacement of P by η and B by π' carries (35) to (22). Elimination of α and δ from (36) and (34) gives

$$\beta C + \epsilon\gamma D \equiv 0, \quad \beta D - \gamma C \equiv 0 \pmod{p},$$
$$C = (l^2 + \epsilon q^2)\rho - (a^2 + \epsilon d^2)U, \quad D = (l^2 + \epsilon q^2)\sigma - (a^2 + \epsilon d^2)V.$$

We shall prove that $C \equiv D \equiv 0$ and hence that $PB \equiv 0 \pmod{p}$. Now

$$\rho = \lambda M + \nu ql - \tau mn + \mu(lm + \epsilon qn),$$

where $\lambda = rs + ru + \epsilon tu$, $\nu = r^2 + 2\epsilon rt - 2\epsilon us - \epsilon u^2$,

$$\tau = (s+u)^2 - 2\epsilon u(s+u) + 2\epsilon rt + \epsilon t^2,$$

$$\mu = r^2 - (s+u)^2 - \epsilon t^2 + \epsilon u^2,$$

$$M = l^2 + ln + \epsilon n^2 - m^2 - mq - \epsilon q^2 \equiv 2(l^2 + ln + \epsilon n^2),$$

since

$$(37) \quad N(P) = l^2 + ln + m^2 + mq + \epsilon n^2 + \epsilon q^2 \equiv 0 \pmod{p}.$$

The final term of ρ is equal to $\mu(U - lq)$. Also, $\nu - \mu = \tau$. Hence

$$\rho \equiv 2\lambda(l^2 + ln + \epsilon n^2) + \tau(lq - mn) + \mu U.$$

Next,

$$(38) \quad \begin{aligned} a^2 + \epsilon d^2 = {} & (r^2 + \epsilon u^2)(l^2 + \epsilon q^2) + \kappa(m^2 + \epsilon n^2) \\ & + 2\epsilon w(mq + ln) + 2\lambda(\epsilon nq - lm), \\ \kappa = {} & (s+u)^2 + \epsilon t^2, \quad w = u(s+u) - rt, \quad \kappa - 2\epsilon w = \tau. \end{aligned}$$

In the resulting expression for C, the sum of the coefficients of $U(l^2 + \epsilon q^2)$ is $\mu - r^2 - \epsilon u^2 = -\kappa$. Combining the resulting term with $-U \cdot \kappa(m^2 + \epsilon n^2)$, we get $-\kappa U(l^2 + \epsilon q^2 + m^2 + \epsilon n^2) \equiv \kappa U(ln + mq)$, by $N(P) \equiv 0$. Combine this with the term $-2\epsilon w(mq + ln)U$ of C; we get $\tau(mq + ln)U$. Adding this to the term $\tau(lq - mn)(l^2 + \epsilon q^2)$ of C, we get $\tau lqN(P) \equiv 0$. The remaining two terms of C have the sum

$$2\lambda[(l^2 + ln + \epsilon n^2)(l^2 + \epsilon q^2) - (\epsilon nq - lm)U] = 2\lambda l^2 N(P).$$

Hence $C \equiv 0$. Next,

$$\begin{aligned} -\sigma = {} & w(l^2 + m^2) + (r^2 + \epsilon u^2 - \epsilon w)q^2 + (\kappa - \epsilon w)n^2 + 2\lambda(lq + mn + qn) \\ & + (w + \kappa - r^2 - \epsilon u^2)nl + (w - \kappa + r^2 + \epsilon u^2)qm, \\ -D = {} & 2\lambda[(l^2 + \epsilon q^2)(lq + mn + qn) + (\epsilon nq - lm)V] \\ & + \kappa[(m^2 + \epsilon n^2)V + (l^2 + \epsilon q^2)(n^2 + nl - qm)] \\ & + w[2\epsilon(mq + ln)V + (l^2 + \epsilon q^2)(l^2 + m^2 - \epsilon n^2 + nl + qm - \epsilon q^2)] \\ & + (r^2 + \epsilon u^2)(l^2 + \epsilon q^2)[V + q^2 - nl + qm]. \end{aligned}$$

The quantities in brackets are respectively equal to

$$qlN(P), \quad (ln - qm)N(P), \quad (l^2 - \epsilon q^2)N(P), \quad 0.$$

Hence $D \equiv 0 \pmod{p}$. This completes the proof in case (i).

(ii) Theorem 9 will be true for P if true for $P_1 = -iP$, $Q_1 = Qi$, for which $Q_1 P_1 = A$. Now P_1 is derived from P by replacing l, m, n, q by m, $-l$, q, $-n$, respectively. Hence the theorem is proved by case (i) unless $l^2 + \epsilon q^2 \equiv 0$, $m^2 + \epsilon n^2 \equiv 0 \pmod{p}$. Then $ln + mq \equiv 0$ by (37). Since expression (38) is not divisible by p, we conclude that $p \neq 2$ and $\epsilon nq - lm \not\equiv 0 \pmod{p}$. But

$$l^2m^2 \equiv (-\epsilon q^2)(-\epsilon n^2),\ lm \equiv \pm \epsilon nq \pmod{p}.$$

Hence the lower sign holds, and $lm \not\equiv 0$. Next,

$$ln \cdot m \equiv -mq \cdot m \equiv \epsilon n^2 \cdot q \equiv n(-lm),\ lnm \equiv 0,$$

whence $n \equiv 0$. Then $m^2 \equiv 0$, contrary to $lm \not\equiv 0$.

LEMMA 10. *If a rational prime p divides AB, $N(A)$ and $N(B)$, where A and B are integral elements, there exist integral elements P, Q, R such that $A = QP$, $B = P'R$, $PP' = p$.*

This is true by Theorem 9 if A is not divisible by p. Next, let $A = Tp$. If $B = pS$, there exists by Theorem 8 an integral element P such that $PP' = P'P = p$, whence Lemma 10 holds for $Q = TP'$, $R = PS$. But if B is not divisible by p, Theorem 7 with $\eta = B$ shows the existence of integral elements P' and R such that $B = P'R$, $P'P = p$. Write $Q = TP'$. Then $A = T \cdot P'P = QP$, $PP' = p$.

THEOREM 10. *All integral solutions of*

$$x^2 + xz + 2z^2 + y^2 + yw + 2w^2 = uv$$

are given by

$$x + yi + zH + wL = fAB,\ u = fN(A),\ v = fN(B),$$

in which f is an arbitrary integer, while the eight coefficients of A and B are any integers without a common factor > 1.

The proof by means of Lemma 10 is exactly like that of Theorem 4.

From Theorem 9 we have at once

LEMMA 11. *If Q, P, B are any integral elements and p is any rational prime such that $PP' = p$ and QPB is divisible by p, either QP or PB is divisible by p.*

From this and Theorem 7 we conclude, as in the proof of Lemma 6 (omitting the sentences employing the unit u):

LEMMA 12. *Any integral element whose norm is divisible by the rational prime p has in common with p a greatest common left divisor which is unique apart from a unit right factor $\pm 1, \pm i$.*

This implies, as in the proof of Theorem 5,

THEOREM 11. *For $\epsilon = 2$, any two integral elements have a greatest common left divisor which is unique up to a unit right factor.*

THE UNIVERSITY OF CHICAGO,
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A Study of the Rational Involutorial Transformations in Space Which Leave a Web of Sextic Surfaces Invariant

By Jesse O. Osborn.

I. Introduction.

1. Consider two spaces (each of three dimensions), (x) and (x'), so related that a general point P_1 of (x) corresponds to a unique point P' of (x'), while a general point P' of (x') corresponds to two points P_1 and P_2 of (x). The correspondence between two such spaces may be expressed in such a way that the planes of the double space (x') correspond to surfaces of some order n in the single space (x) and that three such surfaces intersect in but two variable points. An involution of order 2 in (x) arises through the interchange of the points of associated pairs as P_1 and P_2; that is, P_1 (or P_2) being given, then P' and hence P_2 (or P_1) can be found. This involution will be designated by I.

2. Since the planes of (x') have four linear homogeneous degrees of freedom, the corresponding surfaces in (x) must belong to a web. This web of surfaces will be called ϕ. Each surface of the web must pass through certain fixed elements which together absorb n^3-2 intersections (equivalence) and impose $\frac{(n+1)\ (n+2)\ (n+3)}{6}-4$ conditions on the coefficients of the equation of the surface (postulation). These basis elements consist of curves, simple or multiple on ϕ, of isolated points of various multiplicities, and of points on the basis curves which have greater multiplicity on the surfaces ϕ than the respective curves have. Each of these basis elements and the system as a whole have determinable equivalence and postulation. Two equations are formed. In one the equivalence of all possible combinations of basis elements is equated to the necessary equivalence, and in the other the postulation of all possible combinations of basis elements is equated to the necessary postulation. It is assumed throughout that the basis elements are all independent or in other words that no basis element occurs as a consequence of others. Any solution of these equations, which leads to no geometrical contradiction, furnishes a set of basis elements for ϕ. In determining the equivalence and postulation of the various basis elements the following formulas will be useful.

(1) If a curve C_m is i_1, i_2 and i_3 fold on the three surfaces F_{n_1}, F_{n_2} and F_{n_3} respectively, it is equivalent to E_m points of intersection of the three surfaces, where (r being the rank of C_m)

$$E_m = m(i_2i_3n_1 + i_3i_1n_2 + i_1i_2n_3 - 2i_1i_2i_3) - ri_1i_2i_3.$$

(2) The equivalence of two curves C_m and $C_{m'}$ equals the sum of their separate equivalences minus the quantity $(i'_2i'_3i_1 + i'_3i'_1i_2 + i'_1i'_2i_3 - i'_1i'_2i'_3)s$, where s is the number of points in which C_m and $C_{m'}$ intersect and $i'_1 \leqq i_1$, $i'_2 \leqq i_2$, $i'_3 \lessgtr i_3$.

(3) The postulation of a curve C_m, which is i-fold on a surface F_n is

$$P_m = \frac{i(i+1)}{2}(m(n+2) - \frac{2i+1}{6}(r+2m)).$$

(4) If a point P_{ji} is j_1, j_2 and j_3 fold on the surfaces F_{n_1}, F_{n_2} and F_{n_3} respectively and lies on the curve C_m, then the added equivalence of P_{ji} is given by the formula

$$j_1j_2j_3 - (i_2i_3j_1 + i_3i_1j_2 + i_1i_2j_3 - 2i_1i_2i_3).$$

(5) The added postulation of P_{ji} is

$$\frac{j(j+1)(j+2)}{6} - \frac{i(i+1)}{2}(j+1-\frac{2i+1}{3}).$$

The order and genus of a basis curve i-fold on each surface ϕ are denoted by m_i and p_i respectively, (or where no ambiguity is possible simply by m and p). The number of points of an i-fold basis curve which are j-fold on ϕ is designated by β_{ji}. If $i = 0$, these are isolated basis points and if $j \leq i$ the symbol has no meaning. In the above formulas and in the equations of equivalence and postulation derived from them no account is taken of the possibility that a point at the intersection of two or more basis curves be of greater multiplicity on ϕ than would naturally follow from its being a point of the basis curves. If such account were taken the equations would become extremely complicated and probably few if any essentially new cases would arise.

The method here used was devised by Sharpe and Snyder.* By means

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* F. R. Sharpe and V. Snyder, "Certain Types of Involutorial Space Transformations," *Transactions American Mathematical Society*, vol. 20 (1919), pp. 185-202, and vol. 21 (1920), pp. 52-78. Special cases had been considered earlier by Aschieri, "La trasformazione quadratica doppia di spazio, e la sue applicazione alla geometria dello spazio non euclideo," *Rendiconti del r. Istituto Lombardo*, ser. 2, vol. 14 (1881), pp. 673-683, and vol. 15 (1882) pp. 66-77, 147-154, 247-250; by Reye, "Ueber die Kum-

of it they have studied a large number of the involutorial correspondences that exist when the surfaces ϕ are of order less than six. In this paper the study is extended to the case where these surfaces are sextics. Whenever ϕ is mentioned in the succeeding pages, it refers to this web of sextic surfaces.

3. Besides basis elements any two surfaces ϕ meet in a variable curve C_μ of order μ and genus π, image of a straight line in (x'). $\mu = 36 - \Sigma i^2 m_i$. Assuming the surfaces ϕ to be regular and the basis elements to be independent, π is given as one greater than the arithmetic genus of a general surface of the web. A basis curve C of ϕ has for image in (x') a surface S' of order equal to the number of points in which C_μ meets C. Each point of C has for image in (x') a curve on S' whose order equals the multiplicity of C on ϕ. A basis point of ϕ has for image in (x') a surface of order equal to the multiplicity of the point on C_μ; each direction through the basis point corresponds to a curve on the image surface of order equal to the multiplicity of the basis point on ϕ.

The image in (x') of an arbitrary curve c of order m is a curve c' of order $6m$. For each intersection of c with a basis element k-fold on ϕ, the order of the image curve c' is reduced by k. If c meets enough basis elements of (x) to make $k = 6m$, its image is no longer a curve but simply a point. C_μ meets so many basis elements that its image instead of being of order 6μ is only a line. There may be various curves in (x) that thus correspond to points in (x'), that is each point of such a curve corresponds to a direction through the image point. Such a curve is said to be parasitic.** These curves may be isolated in which case the number that are lines, conics, cubics, $\cdots$ will be designated by a_1, a_2, a_3, $\cdots$, or such a curve may generate a parasitic surface. Every isolated parasitic curve which has been found satisfies the following conditions (1) it is rational; (2) it adds 1 in postulation to the basis elements of ϕ; (3) it adds equivalence 2 to the basis elements of ϕ; (4) In I each point of it corresponds to the whole curve while the curve as a whole is self-corresponding. A parasitic curve which generates a surface may behave in various ways. In this paper will be considered only those cases where the parasitic curve either (a) satisfies the four conditions enu-

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mersche Configuration von sechzehn Punkten und sechzehn Ebenen, *Crelle's Journal*, vol. 86 (1879), pp. 209-212, and "Ueber Strahlensysteme zweiter Classe und die Kummersche Fläche vierter Ordnung mit sechzehn Knotenpunkten," *Crelle's Journal*, vol. 86 (1879), pp. 84-107; and particularly by R. De Paolis, "Le trasformazioni doppie dello spazio," *Memorie della r. Acc. dei Lincei*, ser. 4, vol. 1 (1885), pp. 576-608. The last memoir contains many general formulas.

** This property was first discussed by Schoute, "De la transformation conjuguée dans l'espace," *Association française pour l'avancement des sciences*, Congres de Reims (1880), and more in detail by de Paolis (l. c., p. 595). They use this designation.

merated above, or (b) satisfies (1) and (2) while (3′) it adds equivalence 1 to the basis elements of ϕ so that (4′) in I each point of it corresponds to the whole curve plus another point while the curve as a whole corresponds to this residual point.

In general a point P' of (x') corresponds to two points P_1 and P_2 of (x); for certain positions of P', P_1 and P_2 coincide. The totality of such positions of P' make up the surface of branch points L'; the corresponding points $P_1 \equiv P_2$ make up the surface of coincident points $\overline{K}$, that is $\overline{K}$ taken doubly is the complete image of L'. If a parasitic curve q is self-corresponding in I, then it must lie on $\overline{K}$ and its image Q' must be a point of L'.

The parasitic surfaces generated by curves satisfying (a) or (b) belong to one of two classes. If, in I, the parasitic curve which generates it is self-corresponding or corresponds to a point on the parasitic surface generated, that surface is self-corresponding in I and will be called D. A parasitic surface which is a fixed component of two (or three) linearly independent surfaces ϕ and is such that the pencil (or net) of residual components intersect in basis curves only is self-corresponding in I. All surfaces D that occur in this paper satisfy this condition. It follows that the image in (x') of a surface D is a line d' or a point D'. As D is self-corresponding in I, d' or D' must lie on L' and hence D is a component of $\overline{K}$. After all surfaces D have been taken from $\overline{K}$ the necessary number of times there remains the proper surface of coincidences K. If a parasitic surface is not self-corresponding in I it will be called R. All surfaces R which occur in this study are ruled. Each generator corresponds to a point in (x') and to a point in I; R as a whole corresponds to a curve r' or a point R' in (x') and to a curve r in I. The surface R has the same number of generators in common with K as r' has contacts with L'. These particular generators will be designated as Σt.

The correspondence between (x) and (x') was set up such that the planes of (x') correspond to the web of sextic surfaces through the basis elements of (x), or symbolically, $s'_1 \sim s_6 : \Sigma C_m + \Sigma P$. A plane s_1 of (x) corresponds to a surface of order μ in (x'). Each plane of (x) meets a parasitic curve q of order m in m points; it follows that s'_μ contains Q' m-fold. If Q' describes a curve (in which case q generates a parasitic surface F), then s'_μ contains this curve m-fold. The image in (x) of s'_μ is a surface of order 6μ consisting of the original plane s_1, of F taken m-fold and of a surface s_N. The web of surfaces s_N contains all the basis elements of ϕ to definite multiplicities, contains m-fold each isolated parasitic curve of order m and contains r and Σt if they exist.

4. The existence as well as the nature of an involution depends largely

on the value of π. The order of L' is given by Zeuthen's formula as $2(\pi+1)$; it follows that the order of $\bar{K}$ is $6(\pi+1)$. The multiplicity of each of the basis elements of ϕ on $\bar{K}$ is known. The jacobian J of web ϕ is of order 20. A curve k-fold on ϕ is $4k-1$ fold on J and a point k-fold on ϕ is $4k-2$ fold on J. The jacobian consists of K together with all parasitic surfaces. By a comparison of J and $\bar{K}$ for different values of π one can find what parasitic surfaces may occur. For convenience, at this point only, let C_a, C_b, C_c, $\cdots$ and A, B, C, $\cdots$ designate respectively simple, double, triple, $\cdots$ curves and points, basis on ϕ. Consider

$$\begin{aligned}
\bar{K}_{6(\pi+1)} &: C_a^{\pi+1} + C_b^{2(\pi+1)} + C_c^{3(\pi+1)} + \cdots + A^{\pi+1} + B^{2(\pi+1)} + C^{3(\pi+1)} + \cdots \\
J_{20} &: C_a^{3} + C_b^{7} + C_c^{11} + \cdots + A^{2} + B^{6} + C^{10} + \cdots \\
F_{14-6\pi} &: C_a^{2-\pi} + C_b^{5-2\pi} + C_c^{8-3\pi} + \cdots + A^{1-\pi} + B^{4-2\pi} + C^{7-3\pi} + \cdots
\end{aligned}$$

No non-reducible case enters for which π is zero. If π is 1 or 2 there must be a parasitic surface R and if π exceeds 2 there must be a parasitic surface D. For π equal to 2, ϕ may have no simple basis point for such would necessitate an untrue relation among the multiplicities of such a point on R_2, $\bar{K}$ and J. There can be no simple basis point of ϕ when π is equal to 3. In order that a D_4 exist it must be a component of (at least) two surfaces ϕ. Each basis point simple on ϕ is double on D_4. One of the quadric surfaces which with D_4 makes up a surface ϕ will contain this point while a web of quartic surfaces contains the remaining basis elements of ϕ. Thus any I that might exist can be associated with a web of quartic surfaces. If π equals or exceeds 4, $D_{6\pi-14}$ must be composite in such a way that each component satisfies the conditions which characterize a surface D. For π equal to 4 the only case that has been found to exist is where D_{10} consists of a double quintic surface. This necessitates that ϕ have no point of odd multiplicity and no curve of even multiplicity. No case has been found where π exceeds 4.

The existence of a surface D implies that there is a pencil (or net) of surfaces ψ_i, (such that $D\psi_i$ is a surface ϕ), which corresponds to a pencil (or bundle) of planes in (x'). If ψ_i is known then D is also known. In all cases the basis elements of ψ_i must satisfy the conditions (1) as to equivalence, (2) as to postulation, and (3) as to order of the variable intersection of two ψ_i. These conditions always suffice to characterize the sets of basis elements possible for ψ_i. As an example suppose D is of order 4 and corresponds to a line d'. In this case ψ_i is a pencil of quadric surfaces through certain basis elements of (x). The equivalence and postulation of these basis elements must each equal 8. Two of the quadric surfaces must intersect in basis curves whose orders total 4. A curve of order 4 and genus 1 is the only locus that satisfies these conditions. No set of basis elements is possible for

ψ_i such that D_4 may correspond to a point. As the need arises the criterion set forth in this paragraph will be used as a test for the existence of surfaces D.

When the image of D is $2i$-fold on L', D is an i-fold component of $\overline{K}$. According as D is a component of two or three surfaces ϕ, it is a simple or double component of J.

5. Besides parasitic surfaces there are a_1 lines, a_2 conics, a_3 cubics, $\cdots$ which meet basis elements of (x) in enough points to correspond to points in (x'). These curves are self-corresponding in I. It is possible to get various relations among the a's such that their values may be obtained. (1) from the correspondence between (x') and (x) one knows through what basis curves s_N goes; s_N also has each parasitic curve to multiplicity equal to the order of the curve. Suppose two s_N meet in basis curves whose combined orders, taking account of multiplicity and of r and Σt, equal M. The variable intersection of two s_N is of order N. It follows that $a_1 + 8a_2 + 27a_3 + 64a_4 + \cdots = N^2 - N - M$. (2) The multiplicities of the various basis curves on K are also known. Each parasitic curve is simple on K. Let the combined orders of the basis curves common to K and s_N be M'. When the order of K is N' its variable intersection with an s_N is of order N'. It follows that $a_1 + 4a_2 + 9a_3 + 16a_4 + \cdots = NN' - N' - M'$. (3) The plane s_1 and its image s_N meet in the curve $k_{N'}$, intersection of s_1 and $K_{N'}$, and a curve g of order $N - N'$. The basis and parasitic curves of ϕ pierce s_1 as points of $k_{N'}$ and $g_{N-N'}$. If a curve is m-fold on s_N and m'-fold on $K_{N'}$ then the points in which it pierces s_1 are m' and $m - m'$ fold on $k_{N'}$ and $g_{N-N'}$ respectively. The curves k and g meet in $N'(N - N')$ points which consist of the points in which the basis and parasitic curves pierce s_1 and of η other intersections. Suppose points on the basis curves account for M'' points. Then $2a_2 + 6a_3 + 12a_4 + \cdots + \eta = N'(N - N') - M''$. The number of branch points η in the $(1, 2)$ correspondence between g and its image curve g', the double curve of s'_μ, is given by Zeuthen's formula as $\eta = (2p_g - 2) - 2(2p_{g'} - 2)$. The genus of g designated as p_g is given as $\frac{(m-1)(m-2)}{2} - h$ where the order m and the number of double points h, points of basis or parasitic curves, are known. The order of g' may be found as the number of points in which it pierces a plane s'_1, or what is equivalent as one half the number of non-basis points in which g pierces the surfaces ϕ. The genus $p_{g'}$ of g' is obtained by finding the arithmetic genus of s'_μ by two methods and equating the results. In this equation $p_{g'}$ enters as the only unknown other than the numbers of parasitic curves. Thus η may be written in terms of the a's. (4) A parasitic curve of order m is an m-fold basis curve on the web s_N. One can calculate the postulation necessary to fix the web s_N and also

how much of this arises from the basis elements. Let the difference, a negative number, be designated by $-N''$. The postulation of the parasitic curves must equal $-N''$ or in equation form $a_2 + 4a_3 + 10a_4 + 20a_5 + \cdots = N''$.

Only zero or positive integral values of a are usable. If all correspondences are correct, however, there must be at least one such set of values for the a's which satisfies the above relations. There may be several sets. Each case offers some check that aids one in deciding which solution (or solutions) is correct. By known formulas * the number of fundamental lines may be calculated independently of the above relations; sometimes this may also be done for the number of parasitic conics or other parasitic curves.

6. In proceeding, a systematic and exhaustive search is made for the consistent sets of basis elements of ϕ. This is done by assuming a certain curve (or curves) or set of points to be basis and finding what other basis elements can be added in order that the combination satisfy the equations of equivalence and postulation and present no geometrical contradiction.

II. Discussion of Cases.

A. *Basis Elements Include a Quadruple Curve.*

7. It is impossible that a curve be of multiplicity greater than 4 on a web of sextics. If a curve is 4-fold on ϕ it cannot have order greater than 1. Suppose the basis elements include a 4-fold line C_1. In order that C_μ be non-composite it is necessary that there exists no other basis curve of multiplicity greater than 1. There can exist no basis point, which is not a point of C_1, of multiplicity greater than 2 for in such case the plane of C_1 and the point would be a fixed component of each ϕ. No solution results when one assumes that a point of the 4-fold line is 5-fold on ϕ. With these restrictions the equations of equivalence and postulation are

$$\begin{aligned} 8\beta_{20} + 4\beta_{21} + \beta_{10} &= 52 - 14m + 2p + 11s \\ 4\beta_{20} + 2\beta_{21} + \beta_{10} &= 29 - 6m + p + 4s \qquad m \equiv m_1 \end{aligned}$$

The variable intersection C_μ is of order $\mu = 20 - m$ and has genus $\pi = 1$. A plane through C_1 meets each surface ϕ in a residual conic. These conics can go through not more than two fixed points, that is $s \geq m - 2$. Suppose three points are fixed. We can choose three others on the same plane such that a conic will not go through the six points in question. The images of the three points chosen determine a plane in (x'). The sextic surface, ϕ image of this plane, must go through the six points. The only way this can

* See Pascal's *Repertorium*, 1st edition, vol. 2, p. 231.

happen is for the plane through C_1 to be a component of this particular s_6. This would necessitate that the pencil of planes through C_1 correspond to a pencil of planes in (x'). Such is not the case here. Obviously s cannot exceed m. Three cases may arise according as s equals m, $m-1$, or $m-2$. If $s=m$, C_m must consist wholly of lines and conics skew to each other but coplanar with C_1. If $s=m-1$, C_m must either be rational or composite in such a way that one component is rational and the rest consist wholly of lines and conics skew to each other but coplanar with C_1. There is no restriction on C_m if $s=m-2$. The equations of equivalence and postulation have a large number of solutions. However, most of the cases that might result from these are excluded or reduced birationally by one of the following devices.

(a) Basis elements may occur as a consequence of others. An example of this is where $s=m-2$ and there is a double point. Through this double point two lines can be drawn which meet each ϕ in enough points to become basis. No such case is treated in this discussion.

(b) If there exist as many as two double points on ϕ, either isolated or on C_m, the sextic surfaces which generate the involution may be reduced by a $(2,3)$ birational transformation. The basis elements in (x) of this transformation are C_1 and three basis points, two of which must be double on ϕ.

(c) If C_m has for components at least three skew lines each meeting C_1 the web ϕ may be reduced birationally by a $(3,2)$ transformation. The basis elements in (x) of this transformation are C_1 taken double plus the three skew lines.

(d) If C_m has as a component a line meeting C_1 and there is a double basis point the web of sextic surfaces may be reduced by a $(2,2)$ birational transformation. The basis elements in (x) of this transformation are C_1, the line meeting C_1, and the double point.

The following are the only cases not eliminated by one of the above methods.

	m	p	s	β_{10}	μ	π
A	12	3	10	0	8	1
B	11	2	9	1	7	1
C	10	1	8	2	6	1
D	9	0	7	3	5	1
E	8	−1	6	4	4	1

For discussion the group will first be considered as a whole using s as parameter.

$$m=s+2,\quad \mu=18-s,\quad p=s-7,\quad \beta_{10}=10-s,\quad 6\leq s\leq 10;$$

$$L'_4 \sim K_{12} : C^8{}_1 + C^2{}_{s+2} + (10-s)P^2, \quad J_{20} \equiv K_{12} + R_8 : C^7{}_1 + C_{s+2}.$$

The surface R_8 is generated by the bisecants of C_{s+2} which meets C_1. The image of R_8 is a curve r'_{18-s}. The order of r' is given as the number of points in which it meets a plane s'_1 or, what is equivalent, as the number of lines in which the corresponding s_6 meets R_8. Any plane s_1 meets each generator of R_8; hence the image surfaces s'_{18-s} contain the curve r'. The surface s'_{18-s} has for image in (x) a surface of order $108-6s$ consisting of s_1, R_8 and s_{99-6s} : $C_1^{65-4s} + C_{s+2}^{17-s} + (10-s)P^{18-s}$. The image of s_1 in I is the surface s_{99-6s}. The curve r' has for image in (x) R_8 plus the curve r of order $s^2-33s+267$. The order of r is given as the number of points in which it meets s_1 or, what is equivalent, as the number of lines common to R_8 and s_{99-6s}. There are $36-2s$ lines Σt common to K_{12} and R_8; these correspond to the $36-2s$ contacts of r' with L'_4.

The first three of the devices explained in section 5 suffice to find the numbers of the various parasitic curves. The plane s_1 and its image s_{99-6s} intersect in k_{12} and g_{87-6s}. The image g' of g is of order $\dfrac{s^2-33s+264.}{2}$

The genera of g and g' and the number of branch points η are respectively

$$p_g = \frac{3698-726s+47s^2-s^3}{2} - 3a_3 - 12a_4 - 30a_5 - 60a_6 \cdots$$

$$p_{g'} = \frac{7014-1411s+93s^2-2s^3}{6} - 2a_3 - 8a_4 - 20a_5 - 40a_6 \cdots$$

$$\eta = \frac{s^3-45s^2+644s-2928}{3} + 2a_3 + 8a_4 + 20a_5 + 40a_6 \cdots$$

The relations among the numbers of the various parasitic curves resulting from (1), (2) and (3) of section 5 are respectively

$$a_1 + 8a_2 + 27a_3 + 64a_4 + 125a_5 + 216a_6 + \cdots = 4596 - 848s + 51s^2 - s^3$$

$$a_1 + 4a_2 + 9a_3 + 16a_4 + 25a_5 + 36a_6 + \cdots = 552 - 68s + 2s^2$$

$$2a_2 + 8a_3 + 20a_4 + 40a_5 + 70a_6 + \cdots = \frac{4512-842s+51s^2-s^3}{3}$$

The number of parasitic lines a_1 is calculated independently to be $2(10-s)$, for two lines can be drawn through each basis point which will meet both C_1 and C_{s+2}. These lines are therefore parasitic. For this value of a_1 the above equations have an unique set of solutions. These solutions together with the numerical orders and genera of g and g' and the value of η are given in the following table.

	s	a_1	a_2	a_3	a_4	a_5	a_6	g	g'	p_g	$p_{g'}$	η
A	10	0	0	8	0	0	0	27	17	45	18	20
B	9	2	0	4	4	0	0	33	24	61	25	24
C	8	4	0	2	4	2	0	39	32	79	33	28
D	7	6	0	1	3	3	1	45	41	99	42	32
E	6	8	0	0	4	0	4	51	51	121	52	36

The image in (x') of the line C_1 is a surface S' of order $16-s$ which contains r'; the image in (x) of S'_{16-s} is R_8 plus the surface $S_{88-6s} : C_1^{58-4s} + C_{s+2}^{15-s} + r + \Sigma t + \Sigma a + (10-s)P^{16-s}$. The curve C_{s+2} corresponds to a surface T' of order $32-s$ containing C_1 double; T' in turn corresponds to $R^2{}_8$ plus the surface

$$T_{176-6s} : C_1^{114-4s} + C_{s+2}^{31-s} + r^2 + \Sigma t^2 + \Sigma a + (10-s)P^{32-s}.$$

Each basis point corresponds in (x') to a plane and in I to a sextic surface of web ϕ which has a double point at the original basis point.

The web of sextic surfaces having a quadruple line can be generalized to a web of n-tic surfaces having an $n-2$ fold line. The correspondence is

$$J_{4n-4} : C_1^{4n-9} + C^3{}_{s+2} \sim [K_{2n} : C_1^{2n-4} + C^2{}_{s+2}] + [R_{2n-4} : C_1^{2n-5} + C_{s+2}].$$
$$R_{2n-4} \sim r'_{5n-s-12} \sim R_{2n-4} + r_{20n^2-n(9s+94)+(s^2+21s+111)}.$$
$$s_1 \sim s'_{4n-s-6} : r' \sim s_1 + R + s_{4n^2-ns-8n+3} : C_1^{4n^2-6n-ns+2s+17} + C_{s+2}^{4n-s+3}.$$

The surface R_{2n-4} is generated by the bisecants of C_{s+2} which meets C_1.

B. *Basis Elements Include a Triple Curve.*

8. The order of a curve triple on a web of sextics cannot exceed 1, unless each surface of the web has a fixed component or the web as a whole is reducible. The truth of this is evident upon testing the various possibilities. Assume the basis elements of ϕ to include the triple line C_1. There can be no point, which is not a point of C_1, that is more than 3-fold on ϕ for such would necessitate that the plane of C_1 and this point be a component of each s_6. From the equations of equivalence and postulation it is found that no point of C_1 may be 4- or 5-fold on ϕ. The line C_1 together with three basis points whose combined multiplicities on ϕ equal at least 7 make up the basis system of a $(2,3)$ birational transformation by means of which the order of web ϕ may be reduced. This precludes the possibility of two or more triple points or of one triple point and two or more double points. In continuing there are two cases according as there is or is not a double curve among the basis elements of ϕ.

Suppose there is a double basis curve associated with the triple line C_1.

If the order of this double curve exceeds 1 all cases that arise are at once reducible. Assume there is a double line $\bar{C}_1$. If C_1 and $\bar{C}_1$ intersect a $(2, 2)$ birational transformation whose basis elements are these lines together with a point reduces the order of the surfaces ϕ. There can be no triple basis point which is not a point of $\bar{C}_1$ and no point of $\bar{C}_1$ may be triple if there is a double basis point off this line, for in all such cases the above $(2, 3)$ birational transformation works a reduction in the order of the surfaces ϕ.

Suppose there is no double basis curve but that there is a simple basis curve C_m. If there is a 3-fold basis point then $m - s \leq 3$, for otherwise the plane of C_1 and the triple point would meet each ϕ in basis curves only. Under this condition the equations of equivalence and postulation have no solution.

9. Suppose the basis system of ϕ includes the triple line C_1 and the double line $\bar{C}_1$. Any plane through C_1 meets the surfaces ϕ in residual cubics. Two of these cubics meet in nine points four of which are accounted for at the point where $\bar{C}_1$ pierces the plane. Not more than two of the remaining five points can be fixed. Therefore $m - s_3 \leq 2$ where s_3 is the number of points in which the simple curve C_m meets the triple line. By the same argument it is shown that $m - s_2 \leq 2$, s_2 being the number of points common to C_m and $\bar{C}_1$. The equations of equivalence and postulation are

$$\begin{aligned} 8\beta_{20} + 4\beta_{21} + \beta_{10} &= 48 - 14m + 2p + 8s_3 + 5s_2 \\ 4\beta_{20} + 2\beta_{21} + \beta_{10} &= 26 - 6m + p + 3s_3 + 2s_2. \end{aligned}$$

A number of cases arise from the solutions of the above equations; these are differentiated from one another by the nature of C_m and the distribution of the basis points. The method of treatment is the same in all cases. The group will be treated as a whole using m as parameter.

$$L'_4 \sim K_{12} : C_1^6 + \bar{C}_1^4 + C_m^2 ;$$

$$J_{20} : C_1^{11} + \bar{C}_1^7 + C_m^3 \equiv K_{12} + R_8 : C_1^5 + \bar{C}_1^3 + C_m .$$

$$R_8 \sim r'_{27-m} \sim R_8 + r_{m^2-47m+529}.$$

$$s_1 \sim s'_{23-m} : r' \sim s_1 + R_8 + s_{129-6m} : C_1^{64-3m} + \bar{C}_1^{43-2m} + C_m^{22-m}.$$

10. Suppose the basis elements of ϕ are the triple line C_1, a simple curve C_m and basis points. There can be no basis point of multiplicity greater than 2. The equations of equivalence and postulation are

$$\begin{aligned} 8\beta_{20} + 4\beta_{21} + \beta_{10} &= 104 - 14m + 2p + 8s \\ 4\beta_{20} + 2\beta_{21} + \beta_{10} &= 45 - 6m + p + 3s. \end{aligned}$$

A plane containing C_1 intersects the surface ϕ in residual cubic curves. Two of these curves intersect in nine points. These are points off C_1 where

C_m and C_μ pierce the plane. As C_μ is not rational (genus 4) it must meet the plane in at least two points which are not points of C_1. Therefore $m - s \leq 7$. From the above equations it is found that $2m - 2s - \beta_{10} = 14$. It follows that $m - s = 7$ and $\beta_{10} = 0$. Using s as parameter, $m = 7 + s$, $p = 3s - 3 + 4\beta_{20} + 2\beta_{21}$, $\mu = 20 - s$, $0 \leq s \leq 13$.

There exists a parasitic surface $D_5 : C^2{}_1 + C_m + \Sigma P^2$ which is self-corresponding in I and corresponds to a line d' in (x'). The pencil of planes through C_1 corresponds to the pencil of planes through d'. Designate a plane through C_1 as w and the corresponding plane through d' as w'. The lines of w correspond to cubic curves of genus 0 in w' and conversely. A point of d' corresponds to a curve of order $17 - s$ on D_5. Therefore d' is $17 - s$ fold on each surface s', image of a plane in (x). The surface of branch points L'_{10} has d' 6-fold. The correspondence may be expressed

$$L'_{10} : d'^6 \sim D_5^3 + K_{15} : C_1^9 + C^2{}_{s+7} + \Sigma P^4 ; \; J_{20} \equiv D_5 + K_{15}.$$
$$s_1 \sim s'_{20-s} : d'^{17-s} \sim s_1 + D_5^{17-s} + s_{34-s} : C_1^{26-s} + C^3{}_{s+7} + \Sigma P^6 .$$

The number of the various parasitic curves may be obtained by methods previously explained. A plane s_1 and its image s_{34-s} in I meet in k_{15} and g_{19-s}. The image g' of g is of order $28 - 2s$. The genera of g and g' and the number of branch points in the $(1, 2)$ correspondence between them are

$$p_g = (17 - s) - 3a_3 - 12a_4 - 30a_5 - 60a_6 \cdots$$
$$p_{g'} = 0 - 2a_3 - 8a_4 - 20a_5 - 40a_6 \cdots$$
$$\eta = (36 - s) + 2a_3 + 8a_4 + 20a_5 + 40a_6 \cdots$$

Consideration of the intersection of two s_{34-s}, of an s_{34-s} and K_{15}, and of k_{15} and g_{19-s} leads respectively to the equations

$$a_1 + 8a_2 + 27a_3 + 64a_4 + 125a_5 + 216a_6 + \cdots = 383 - 24s$$
$$a_1 + 4a_2 + 9a_3 + 16a_4 + 25a_5 + 36a_6 + \cdots = 219 - 12s$$
$$2a_2 + 8a_3 + 20a_4 + 40a_5 + 70a_6 + \cdots = 82 - 6s$$

The only solution of the above equations is

$$a_1 = 55; \quad a_2 = 41 - 3s; \quad a_i = 0, \; i > 2.$$

A parasitic line either meets C_1 once and C_{s+7} three times or goes through a double basis point and meets both C_1 and C_{s+7}; thus the number a_1 may be calculated independently.

The line C_1 corresponds in (x') to a surface S'_{18-s} having d' $15 - s$ fold, and in I to a surface S_{33-s}. The curve C_{s+7} corresponds in (x') to a surface T'_{64-3s} having d' $57 - 3s$ fold, and in I to a surface T_{99-3s}.

C. *Basis Elements Include a Double Curve.*

11. Suppose the basis elements of ϕ consist of a double curve, a simple

curve and points of various multiplicities. The order of the double curve cannot exceed 8. If the order is 8, the genus cannot be less than 8 for if such were the case the postulation of C_8 alone would be too great for it to be double on a sextic surface. There remain two possible values for the genus, namely 8 and 9. The curve C_8 $(p = 8)$ is a partial intersection of a quadric and a quintic surface and has a single infinity of lines meeting it in five points, the generators of the quadric surface. Thus each sextic surface has a fixed quadric as a component and the involution is reduced to one determined by quartic surfaces. Similarly, C_8 $(p = 9)$ is the complete intersection of a quadric and a quartic surface; each sextic surface again has a fixed quadric as a component. If the order of the double curve is 7 its genus must be at least 6 for otherwise the postulation of C_7 alone is too great for it to be double on a sextic surface. The only remaining case C_7 $(p = 6)$ is a partial intersection of a quadric and a quartic surface and hence has a single infinity of quadri-secants. This quadric must be a component of each sextic surface. If the order of the double curve is 6 the genus cannot be less than 3, otherwise the postulation of C_6 alone would be too great for it to be double on a sextic surface. The curve C_6 $(p = 3)$ has equivalence and postulation of 26 and 16 respectively on cubic surfaces. The residual intersection of two such cubic surfaces is a space cubic. A birational $(3, 3)$ transformation will reduce all s_6 on which C_6 $(p = 3)$ is a double curve to quadric surfaces.

12. If the double basis curve is of order 6 and genus 4 it is necessary that $s = 2m$ and that no basis point of multiplicity greater than 2 exist. The equations of equivalence and postulation reduce to

$$4\beta_{20} + 2\beta_{21} = 10 - 2m + p$$

The following solutions exist. (There are two other geometrically consistent solutions, $m = 4$, $p = -2$ and $m = 3$, $p = -2$, $\beta_{21} = 1$. However, because of the composite nature of C_m and of the resulting surfaces ϕ, a discussion of these is outside the scope of this paper.)

	A	B	C	D	E	F	G	H	I	J
m	6	5	4	3	3	2	2	1	1	1
p	2	0	0	0	0	0	0	0	0	0
s	12	10	8	6	6	4	4	2	2	2
β_{20}	0	0	0	0	1	1	0	2	0	1
β_{21}	0	0	1	2	0	1	3	0	4	2
μ	6	7	8	9	9	10	10	11	11	11
π	2	2	2	2	2	2	2	2	2	2

The curve C_6 determines a quadric surface D_2 which corresponds to a point

in (x'). In I each point of D_2 corresponds to the whole surface. The characteristics of the above cases are alike; they will be treated as a group using m as parameter.

$$s_1 \sim s'_{12-m} \sim s_1 + D_2^2 + s_{67-6m} : C_6^{22-2m} + C_m^{12-m} + \text{parasitic curves.}$$
$$L'_6 \sim D_2 + K_{16} : C_6^5 + C_m^3; \quad J_{20} \sim D_2^2 + K_{16}$$
$$C_6 \sim S'_{24-2m} \sim D_2^6 + S_{132-12m} : C_6^{48-4m} + C_m^{24-2m}.$$
$$C_m \sim T'_{22-2m} \sim T_{132-2m} : C_6^{44-4m} + C_m^{22-2m}.$$

The number of the various parasitic curves may be obtained from a consideration of the following relations. The double curve g' of s' is of order $\frac{m^2 - 21m + 102}{2}$ and its image curve g is of order $51 - 6m$.

$$a_1 + 8a_2 + 27a_3 + 64a_4 + 125a_5 + \cdots = 1518 - 414m + 36m^2 - m^3$$
$$a_1 + 4a_2 + 9a_3 + 16a_4 + 25a_5 + \cdots = 395 - 72m + 3m^2$$
$$\eta + 2a_2 + 6a_3 + 12a_4 + 20a_5 + \cdots = 306 - 63m + 3m^3$$

The characteristics of any particular case may be obtained by substituting the proper value of m in the above relations.

13. It is impossible to have a curve of order 5 and genus less than 1 double on sextic surfaces for the postulation is too great. The curve C_5 $(p = 1)$ has equivalence and postulation of 25 and 15 respectively on cubic surfaces. Choose a point P, either an isolated basis point or a point of the simple basis curve, making the combined equivalence and postulation of C_5 and P 26 and 16 respectively. A birational transformation will reduce the sextics of web ϕ to cubic surfaces.

The curve C_5 $(p = 2)$ has a quadric surface R_2 as surface of trisecants. Each generator of R_2 corresponds to a point in (x'); R_2 as a whole corresponds to a curve r'_2. The variable intersection of two surfaces ϕ cannot meet R_2 in a point other than a point of C_5 for such would necessitate that all lines in (x') meet r'_2. Since C_5 lies on a quadric surface, C_m can meet it in not more than $2m$ points. It follows that $s = 2m - k$ where k is zero or a positive integer. But if k has a value other than zero all cases that may arise are but special types of cases in which k equals zero, for through each point in which C_m meets R_2 and not C_5 there is a generator of R_2 which is basis on ϕ. The proper C_m plus this line (or these lines) make up a new simple basis curve for which k equals zero. Thus there is no restriction in making $s = 2m$. For $\pi = 2$ there can be no simple basis point. It follows from the equations of equivalence and postulation that there can be no point of mul-

tiplicity greater than 2 on ϕ. The equations of equivalence and postulation reduce to

$$4\beta_{20} + 2\beta_{21} = 14 - 2m + p.$$

Before enumerating the solutions of this equation it is possible to discard a great number as leading either to a geometrical contradiction or to cases which are reducible. From the postulation of C_5 plus C_m it is found that if $m + 4\beta_{20} + 2\beta_{21} \leq 11$ a pencil of cubic surfaces will go through C_5 and C_m. However, this is contradictory if m exceeds 4. Other cases may be discarded because a cubic surface will contain C_5 and C_m while a web or more of cubic surfaces goes through the remaining basis elements. There is no need to consider any case where C_m is composite in such a way that one component is a line meeting C_5 twice. All such cases are reducible by a $(3, 3)$ birational transformation whose basis elements are C_5 and this line. The remaining solutions are given in the table.

	A	B	C	D	E	F	G	H
m	9	8	7	6	7	6	5	4
p	4	2	0	-2	2	0	0	0
s	18	16	14	12	14	12	10	8
β_{10}	0	0	0	0	0	0	0	0
$2\beta_{20} + \beta_{21}$	0	0	0	0	1	1	2	3
μ	7	8	9	10	9	10	11	12
π	2	2	2	2	2	2	2	2

The curve r'_2 corresponds to R_2 plus a curve r of order $25 - 2m$. This order may be obtained as previously explained or by considering the non-basis intersection of an s_4 and an s_{10}, images respectively of a plane and a quadric containing r'_2. Besides basis elements K_{18} and R_2 have in common six generators $6t$ of the latter; these correspond to the six contacts of r'_2 and L'_6. Using m as parameter the correspondence is

$$s_1 \sim s_{16-m} \sim s_1 + R_2 + s_{93-6m} : C_5^{31-2m} + C_m^{16-m} + r + 6t.$$

$$L'_6 \sim K_{18} : C_5^6 + C_m^3; \quad J_{20} \equiv K_{18} + R_2.$$

$$C_5 \sim S'_{32-2m} \sim R^3 + S_{186-12m} : C_5^{62-4m} + C_m^{32-2m}.$$

$$C_m \sim T'_{30-2m} \sim T_{180-12m} : C_5^{60-4m} + C_m^{31-2m}.$$

The number of the various parasitic curves may be determined by use of the following relations. The curve g is of order $75 - 6m$ and its image curve g' is of order $\dfrac{m^2 - 29m + 200}{2}$.

$$p_\sigma = \frac{2402 - 548m + 41m^2 - m^3}{2} - 3a_3 - 12a_4 - 30a_5 - 60a_6 \cdots$$

$$p_{\sigma'} = \frac{4476 - 1057m + 81m^2 - 2m^3}{6} - 2a_3 - 8a_4 - 20a_5 - 40a_6 \cdots$$

$$\eta = \frac{m^3 - 39m^2 + 470m - 1740}{3} + 2a_3 + 8a_4 + 20a_5 + 40a_6 \cdots$$

$$a_1 + 8a_2 + 27a_3 + 64a_4 + 125a_5 + \cdots = 3720 - 744m + 48m^2 - m^3$$

$$a_1 + 4a_2 + 9a_3 + 16a_4 + 25a_5 + \cdots = 720 - 96m + 3m^2$$

$$2a_2 + 8a_3 + 20a_4 + 40a_5 + \cdots = \frac{3540 - 731m + 48m^2 - m^3}{3}$$

For each of the above cases one can substitute in these equations the proper value for m and obtain the details for that particular case.

14. When the double curve is a quartic it is impossible that its genus be -3 for then C_μ would be composite. If the genus is -2, C_4 must consist of a conic and two lines mutually skew. A line in the plane of the conic meeting both the double basis lines is itself basis. The conic and these three lines form the basis elements of a $(3, 4)$ birational transformation by means of which ϕ may be reduced to a web of quintics. The curve C_4 $(p = -1)$ together with two points furnish the basis elements for a $(3, 5)$ birational transformation which reduces the web of sextic surfaces to a web of quartic surfaces. If C_4 is rational web ϕ is reducible provided there exists a basis point of multiplicity three or more on ϕ but not a point of C_4. A line may be drawn through this basis point which will meet C_4 twice and be therefore itself basis. A birational transformation whose basis elements are C_4, this line and a basis point reduces the sextics to surfaces of lower order. This same transformation works a reduction, even if there is no triple basis point, if C_m is composite in such a way that one component of it is a line which meets C_4 twice. For this reason it is necessary to impose the restriction that $m \geq 2 - 2p$. As C_4 $(p = 0)$ lies on a quadric surface and as C_m cannot be on the same quadric $s \leq 2m$. As previously explained it is no restriction to make $s = 2m$. There can be no simple basis point when $\pi = 2$; from substituting this fact in the equations of equivalence and postulation it follows that no point of C_4 can be of greater multiplicity on ϕ than is the curve as a whole. If $m + 4\beta_{20} + 2\beta_{21} \geq 14$ a pencil of cubics can go through C_4 and C_m but this is contradictory if m exceeds 5. The only remaining cases are given in the table below.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
m	13	12	11	10	9	8	11	10	9	8	7	9	8	7	6	7	6	5	5	4	3	2
p	8	6	4	2	0	−2	6	4	2	0	−2	4	2	0	−2	2	0	0	2	0	0	0

s	26	24	22	20	18	16	22	20	18	16	14	18	16	14	12	14	12	10	10	8	6	4
$2\beta_{20}+\beta_{21}$	0	0	0	0	0	0	1	1	1	1	1	2	2	2	2	3	3	4	5	5	6	7
μ	7	8	9	10	11	12	9	10	11	12	13	11	12	13	14	13	14	15	15	16	17	18
π	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

In all the above cases the correspondence follows the same law. The curve C_4 $(p=0)$ determines a quadric surface R_2 which has for image in (x') a quartic curve r'_4 and in I a curve r of order $66-4m$. The correspondence is

$$L'_6 \sim K_{18} : C_4^6 + C_m^3; \quad J_{20} \equiv K_{18} + R_2; \quad (K_{18}, R_2) = 12t.$$
$$s_1 \sim s'_{20-m} \sim s_1 + R_2 + s_{117-6m} \cdot C_4^{39-2m} + C_m^{20-m}.$$
$$C_4 \sim S'_{40-2m} \sim R_2^3 + S_{234-12m} : C_4^{78-4m} + C_m^{40-2m}.$$
$$C_m \sim T'_{38-2m} \sim T_{228-12m} : C_4^{76-4m} + C_m^{40-2m}.$$

Suppose C_4 is of genus 1. If there is a point of multiplicity greater than 2 on ϕ which is not a point of C_4 the web of sextics can be reduced birationally by a $(3,4)$ transformation. The basis elements of this transformation are C_4, any line meeting C_4 once and the triple point. For $\pi=3$ it is necessary that $\beta_{10}=0$ and from the equations of equivalence and postulation it follows that $\beta_{j2}=0$ for all j. If there exists a double basis point, C_4 plus this point determines a quadric surface while a web of quartic surfaces goes through the remaining basis elements. In order that each s_6 shall not have as a component a fixed cubic surface through C_4 and C_m it is necessary that m be less than 11. The remaining cases are given in the table. $(\mu = 20-m,\ \pi=3)$

	A	B	C	D	E	F
m	10	9	8	7	6	5
p	9	7	5	3	1	-1
s	14	12	10	8	6	4

A quartic surface D_4 is fixed by C_4 and C_m while a pencil of quadric surfaces goes through C_4 alone. The image of D_4 in (x') is a line d' and in I it is self-corresponding. Each point of d' corresponds to a curve of order $16-m$ on D_4. The correspondence is

$$L'_8 \sim K_{16} : C_4^6 + C_m^2 + D_4^2; \quad J_{20} \equiv K_{16} + D_4.$$
$$s_1 \sim s_{20-m} \sim s_1 + D_4^{16-m} + s_{55-2m} : C_4^{24-m} + C_m^4.$$
$$C_4 \sim S'_{38-2m} \sim D_4^{30-2m} + S_{108-4m} : C_4^{47-2m} + C_m^8.$$
$$C_m \sim T'_{42-2m} \sim D_4^{36-2m} + T_{108-4m} : C_4^{48-2m} + C_m^7.$$

There is no parasitic curve of order greater than 3. The number of fundamental lines, conics and cubics respectively is $27+m$, $36-2m$ and $13-m$.

15. Suppose the double curve is a cubic. If the genus is -2, C_3 determines a quadric surface R_2. As previously shown it is no restriction to make $s = 2m$. The postulation is such that if there is a basis point of multiplicity greater than 2, R_2 is a fixed component of each s_6. From the equations of equivalence and postulation one finds that $\beta_{10} = 0$ and $p = 2m - 22 + 4\beta_{20} + 2\beta_{21}$. The correspondence in all cases that arise follows the same law. The general characteristics are

$$L'_6 \sim K_{18} : C_3^6 + C_m^3 ; \quad J_{20} \equiv K_{18} + R_2 ; \quad (K_{18}, R_2) = 18t.$$
$$s_1 \sim s'_{24-m} \sim s_1 + R_2 + s_{141-6m} : \quad C_3^{47-2m} + C_m^{24-m}.$$
$$R_2 \sim r'_6 \sim R_2 + r_{123-6m}.$$
$$C_3 \sim S'_{48-2m} \sim R_2^3 + S_{282-12m} ; \quad C_m \sim T'_{46-2m} \sim T_{276-12m}.$$

Suppose the double basis curve is of order 3 and genus -1, that is consists of a conic and a line skew to the conic. If there is a basis point of multiplicity 3 or more, which is not a point of the conic, the surfaces are reducible. First, if the triple point is in the plane of the conic this plane is a component of each s_6. Second, if the triple point is not in the plane of the conic a $(2, 2)$ birational transformation will reduce ϕ to a web of quintic surfaces. The double conic and the triple point each taken simple are the basis elements of this transformation. The equations of equivalence and postulation have no consistent solution when a point of the conic is triple or quadruple on ϕ. There remain therefore only basis points whose multiplicities on ϕ do not exceed 2. The genus of C_μ is 3. It is necessary that there exists a surface D_4. This implies that the associated web of quadrics have in common a curve of order 4 and genus 1 (section 4). This is impossible here.

If the double cubic be of genus 0 no I has been found to exist. Contradictory relations arise with regard to parasitic surfaces. There is, however, an involution J of order 2 associated with every pencil ψ of web ϕ. If a point P_1 is fixed it determines a surface ψ_1 while the line through P_1 and meeting C_3 $(p = 0)$ twice meets ψ_1 in but one other point Q_1. The involution J is between such pairs of points as P_1 and Q_1. The congruence of lines which meets C_3 twice is birationally equivalent to a bundle of lines through some point A. Thus J can be birationally reduced to a monoidal involution and as such has already been studied.

When the double cubic has genus 1 no involutorial correspondence has been found.

16. It will save repetition to make a digression here. In the last section no I could be found when C_3 was of genus 0 or -1. The reason given

was that contradictory relations existed with respect to parasitic surfaces. The same difficulty enters in all cases yet to be considered. Specifically the complication is that π is so great and therefore the orders of L' and $\overline{K}$ so large that no parasitic surface or surfaces D can be found which satisfy a consistent relation among $\overline{K}$, K, D and J. It is needful here to restate the conditions necessary for the existence of a surface D.

If a surface D corresponds to a line d' there must exist a pencil of surfaces S which corresponds to the pencil of planes through d' and are such that D together with an S form a sextic of web ϕ. Two of the planes in (x') meet in no point other than d'; it follows that the two corresponding surfaces S meet either in basis curves alone or contain besides basis curves some fixed curve on D. There exist therefore three relations among the curves through which each S goes, namely, (1) their combined postulation must be such that there is a pencil of surfaces S; (2) their combined equivalence must be such that three S meet in no variable point; (3) two S can meet in no variable curve.

If D corresponds to a point D' the above relations become (1) the combined postulation of the curves through which S goes must be such that there is a net of surfaces S; (2) their combined equivalence must be such that three surfaces S meet in no variable point; (3) two S meet in a variable curve of order $i^2 - j$ where S is of order i and j is the order of the image of a line through D'.

Whenever a case is excluded because of contradictory relations with respect to parasitic surfaces it means that no D can be found that will satisfy the necessary conditions (1) among $\overline{K}$, K, D and J, and (2) as to containing the necessary basis elements.

17. Suppose the double curve is a proper conic and consider the following cases. (1) If there is a basis point of multiplicity greater than 2 which is not in the plane of the conic a $(2, 2)$ birational transformation will reduce web ϕ. (2) If there is a basis point of multiplicity greater than 2 in the plane of the conic but not a point of the conic this plane is a component of each ϕ. (3) In all other cases a quartic surface will contain the conic single fold, the simple curve and all the basis points while more than a web of quadric surfaces will go through the conic alone.

Suppose the double basis curve consists of two skew lines. If there are three basis points whose combined multiplicities equal or exceed 9 any I that might exist is reducible by a $(2, 3)$ birational transformation whose basis elements are one of the double lines and the three points. If there is a quadruple basis point all solutions of the equations of equivalence and postulation are such that the simple curve must meet the plane of this point and one

of the double lines in at least two points; thus any I that might result may either be generated by surfaces of order lower than 6 or disintegrates. In all other cases the value of π is so great as to introduce inconsistent relations among the parasitic surfaces. As with C_3 $(p = 0)$ there exists in this case an involution J defined in the same way.

Assume that the double curve is a straight line. In case there are three basis points whose combined multiplicities equal at least 9 the $(2, 3)$ transformation of the preceding paragraph again applies. In all other cases no parasitic surfaces exist which satisfy the necessary conditions.

If one sets up the equations of equivalence and postulation so as to have only a simple basis curve together with basis points or to have basis points alone no I results.

18. Although it was shown that involutions exist in cases where there are double basis points among the basis elements no account was taken of this fact in tabulating the previous correspondences. The entire revision can now be conveniently effected. An isolated double point has for image in (x') a quartic surface which returns to (x) as a surface of order 24; a point of a simple basis curve which is double on all surfaces ϕ has for image in (x') a cubic surface which returns to (x) as a surface of order 18. These surfaces may or may not have parasitic surfaces as components. If there is a double basis point among the basis elements of ϕ, C_μ meets some or all the basis curves in a less number of points than if no double basis point existed. Thus the orders of $S, T, \cdots$ are accordingly decreased. The order of the jacobian J of web s_N is independent of the number of double basis points and if there is no such point is made up of $S, T, \cdots$ and R. If there is a double basis point the image in I of this point is included in J.

III. CONCLUSION.

19. This study has attempted to classify but a limited number of the $(2, 1)$ correspondences associated with a web of sextic surfaces. Within the bounds set in the Introduction, it is believed that this classification is complete. The work done here has also suggested avenues for future investigation, for example, a more detailed discussion of certain irregular cases, or, an exposition of the properties of the various types of basic and parasitic elements.

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On the Reduction of Differential Parameters in Terms of Finite Sets, With Remarks Concerning Differential Invariants of Analytic Transformations.*

By Oliver E. Glenn.

Part 1.

I. Introduction. The first investigator to give comprehensive thought to differential invariants was perhaps E. B. Christoffel (1869),† although Gauss‡ (1827) previously identified an important invariant and Lamé had employed special cases. The methods in the subject that seem to be the most fundamental in the developments of the theory of relativity are those of Christoffel as reinforced by the researches of Ricci and Levi-Civita,** in particular, the latter authors developed a general process for the identification of complete systems. The theory as due to Sophus Lie has been applied by Haskins § to determine the number of functionally independent invariants of any order. Maschke,§§ in several papers, established symbolical methods.

The object of the present paper is to secure, within certain domains, not only the number of fundamental parameters, but also their explicit expressions. Where the problem of the present paper coincides with that of the theory of quadratic differential forms in n variables: (The case $n = 2$, $m = 2$), we obtain linear relations which connect the fundamental jacobian parameters here employed with Christoffel's triple index symbols (¶ VIII).

II. Didactic Resumé. In a paper which was published in 1921 * I established a number of results to which reference is here made in brief. Let

$$F = (a_0, \cdots, a_m)(dx_1, dx_2)^m = (df)^m = (d\phi)^m, \text{ (Maschke)}$$

be subjected to arbitrary functional transformation

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* Presented to the American Mathematical Society; February and September, 1923.

† Crelle, *Journal für Math.*, Vol. 70 (1869), p. 46.

‡ *Disquisitiones generales circa superficies curvas* (1827), Werke, Vol. 4, p. 217.

** *Mathematische Annalen*, Vol. 54 (1901), p. 125.

§ *Transactions Amer. Math. Soc.*, Vol. 3 (1902), pp. 71, 91; vol. 5 (1904), pp. 167, 192.

§§ *Transactions Amer. Math. Soc.*, Vol. 1 (1900), p. 197; Vol. 4 (1903), p. 445.

* *Proceedings of the National Academy of Sciences*, Vol. 7 (1921), p. 276; *Annals of Math.*, (2), Vol. 23 (1921), p. 16.

$$(1)\qquad T : x_i = x_i(y_1, y_2)\,;\ dx_i = \frac{\partial x_i}{\partial y_1}\,dy_1 + \frac{\partial x_i}{\partial y_2}\,dy_2\ \ (i = 1, 2).$$

The two differential covariants whose roots are the respective poles of dx_i are

$$(2)\qquad df_{\pm 1} = 2\,\frac{\partial x_2}{\partial y_1}\,dx_1 + \left(\frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_1} \pm \Delta\right) dx_2,$$

and

$$(3)\qquad df'_{\pm 1} = \rho_{\pm 1}^{-1}\, df_{\pm 1},$$

where

$$(4)\qquad \Delta = \left[\left(\frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_1}\right)^2 + 4\,\frac{\partial x_1}{\partial y_2}\,\frac{\partial x_2}{\partial y_1}\right]^{\frac{1}{2}},$$

$$\rho_{\pm 1} = \frac{1}{2}\left(\frac{\partial x_1}{\partial y_1} + \frac{\partial x_2}{\partial y_2} \pm \Delta\right),\ \rho_{+1}\rho_{-1} = D,$$

D being the determinant of the transformation on the differentials. We associate only like signs and employ primes to indicate functions of y_1, y_2. We may also make the abbreviations

$$(5)\qquad f_1 = \partial f/\partial x_1,\quad f_2 = \partial f/\partial x_2,\quad a_1 = \partial x_1/\partial y_1,\quad a_2 = \partial x_1/\partial y_2,$$
$$\beta_0 = \partial x_2/\partial y_1,\quad \beta_1 = \partial x_2/\partial y_2.$$

If F be transformed by the inverse of (2), not only the new differential variables but also the new coefficients are invariantive under the transformation of F by (1), viz.,

$$(6)\qquad \phi'_{m-2i} = \rho_{+1}^{m-2i}\, D^i\, \phi_{m-2i} = Q^{(i)}\phi_{m-2i}\ \ (i = 0, \cdots, m),$$

in which

$$(7)\qquad \phi_{m-2i} = [(\gamma_1 - \Delta)f_1 - 2\beta_0 f_2]^{m-i}[-(\gamma_1 + \Delta)f_1 + 2\beta_0 f_2]^i \times (-4\beta_0\Delta)^{-m},\ \ (\gamma_1 = \beta_1 - a_1).$$

It is evident that

$$(8)\qquad \frac{\partial}{\partial y_1} = a_1\frac{\partial}{\partial x_1} + \beta_0\frac{\partial}{\partial x_2} = \Omega_1,\quad \frac{\partial}{\partial y_2} = a_2\frac{\partial}{\partial x_1} + \beta_1\frac{\partial}{\partial x_2} = \Omega_2,$$

hence, if we write

$$(U, V) = \frac{\partial U}{\partial x_1}\frac{\partial V}{\partial x_2} - \frac{\partial U}{\partial x_2}\frac{\partial V}{\partial x_1},$$

we obtain

$$\Psi^{(0,1)\prime}_{m-2i} = (\phi'_{m-2i}, Q^{(i)\prime}) = Q^{(i)} D(\phi_{m-2i}, Q^{(i)}) = Q_1{}^{(i)}\,\Psi^{(0,1)}_{m-2i},$$
$$(i = 0, \cdots, m),$$

$$(9)\quad \Psi^{(0,\,2)\prime}_{m-2i} = ((\phi'_{m-2i}, Q^{(i)\prime}), Q_1^{(i)\prime}) = Q_2^{(i)}((\phi_{m-2i}, Q^{(i)}), Q_1^{(i)}),$$
$$Q_2^{(i)} = Q_1^{(i)}D,$$
$$\Psi^{(0,\,3)\prime}_{m-2i} = (((\phi'_{m-2i}, Q^{(i)\prime}), Q_1^{(i)\prime}), Q_2^{(i)\prime})$$
$$= Q_3^{(i)}(((\phi_{m-2i}, Q^{(i)}), Q_1^{(i)}), Q_2^{(i)}), Q_3^{(i)} = Q_2^{(i)}D,$$

. . . . . .

and in general,

$$(10)\qquad \Psi^{(0,\,n)\prime}_{m-2i} = \rho_{+1}^{m-2i} D^{i+n} \Psi^{(0,\,n)}_{m-2i}\,, \qquad (n = 0, \cdots, \infty),$$

the expression $\Psi^{(0,\,0)}_{m-2i}$ being, by definition, equal to ϕ_{m-2i}.

III. DOMAINS OF RATIONALITY. When we speak of reducibility of concomitants and, more generally, of functional dependence, the domain of the operations must be specified. In view of the linearity of the parameter ϕ_{m-2i} of (7) in $a_0, \cdots, a_m$ every differential parameter, which is associated with F in consequence of (1), is reducible in terms of the concomitants

$$(11)\qquad \phi_{m-2i},\ df_{\pm 1}.$$

Accordingly, these form a fundamental system, but only within the domain $R(1, T^{(1)}, \Delta)$ which is rational in the first derivatives of the functions $x_i(y_1, y_2)$, in the transformations T and in the quadratic irrationality Δ. A more general domain which is to concern us is $R(1, T^{(n)}, \Delta)$ whose essential defining quantities are the derivatives of $x_i(y_1, y_2)$ up to the n-th and also Δ. The domain $R(1, T^{(n)}, 0)$ is of the same description aside from the omission of Δ, whereas $R(1, 0, 0)$ is the field that is free from the coefficients of the transformation or their derivatives. This latter domain is the one to which the investigations of former writers relate.

It is well to be reminded that while there is a tendency towards a condition that the invariant elements (11) should be concomitants of only one transformation, yet there are two circumstances which make them invariants of an infinitude. The derivative of an arbitrary function being arbitrary, the functions x_i in dx_i of T can differ from the $x_i(y_1, y_2)$ of T. The former need only be such that they have the same four partial derivatives, respectively, as the latter. Secondly the derivatives β_1, a_1 enter these elements only in the combination β_1—a_1, and hence the value of one derivative remains independent through a relation

$$(12)\qquad \frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_1} = \xi,$$

with ξ assigned. Apart from these exceptions, however, the invariant ele-

ments depend upon a restricted set of transformations and it is only when the concomitants simplify from the domains $R(1, T^{(n)}, \Delta)$, $R(1, T^{(n)}, 0)$, to $R(1, 0, 0)$. as extensive classes do, that we can refer to the system of differential parameters as being concomitants of F under all transformations (arbitrary) of T. Further comment in this connection is made in Part II of this paper, where certain particularizations are made in connection with the derivatives a_1, a_2, β_0, β_1.

If, for particular functions x_1, x_2, the expressions (2) require an integrating factor μ to render them exact differentials, then df_{+1}, df'_{+1} have the same integrating factor μ.

IV. TRANSVECTION GENERALIZED. Let I be any differential parameter:

$$I' = RI. \tag{13}$$

Assume that both R and I are functions of x_1, x_2 and that they are differentiable. Also let u_1 be any function that satisfies the relation $u'_1 = u_1$ under (1) and which is functionally dependent upon R;

$$(u_1, R) \equiv \begin{vmatrix} \dfrac{\partial R}{\partial x_1}, & \dfrac{\partial R}{\partial x_2} \\ \dfrac{\partial u_1}{\partial x_1}, & \dfrac{\partial u_1}{\partial x_2} \end{vmatrix} = 0. \tag{14}$$

If we apply the relations (8) we therefore obtain

$$\begin{vmatrix} \dfrac{\partial I'}{\partial y_1}, & \dfrac{\partial I'}{\partial y_2} \\ \dfrac{\partial u'_1}{\partial y_1}, & \dfrac{\partial u'_1}{\partial y_2} \end{vmatrix} \equiv \begin{vmatrix} I\Omega_1 R + R\Omega_1 I, & I\Omega_2 R + R\Omega_2 I \\ \Omega_1 u_1, & \Omega_2 u_1 \end{vmatrix} \tag{15}$$

$$= RD \begin{vmatrix} \dfrac{\partial I}{\partial x_1}, & \dfrac{\partial I}{\partial x_2} \\ \dfrac{\partial u_1}{\partial x_1}, & \dfrac{\partial u_1}{\partial x_2} \end{vmatrix}. \tag{16}$$

Thus, under the hypotheses, (I, u) is an invariant and its relation of concomitance can be iterated after the plan of (9):

$$\begin{aligned} \Psi^{(c,\,1)\prime} &= (I', u'_1) = RD(I, u_1), \\ \Psi^{(c,\,2)\prime} &= ((I', u'_1), u'_2) = RD^2((I, u_1), u_2), \\ \Psi^{(c,\,3)\prime} &= (((I', u'_1), u'_2), u'_3) = RD^3(((I, u_1), u_2), u_3), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Psi^{(c,\,r)\prime} &= (\cdots(I', u'_1), u'_2), \cdots, u'_r) = RD^r(\cdots(I, u_1), u_2), \cdots, u_r). \end{aligned} \tag{17}$$

The functions u_r $(r = 1, 2, \cdots)$ are any functions for which $u'_r = u_r$ and which satisfy the following relations of functional dependence by hypothesis: $(u_r, RD^{r-1}) = 0$. Note that one solution (identical) of the latter linear partial differential equation is $u_r = RD^{r-1}$. If I is a differential parameter of order t in regard to derivatives of the coefficient functions of F; $a_0, \cdots, a_m$, then, $\Psi^{(c,r)}$ is of the order $r + t$.

The best known method of deriving, from differential parameters involving derivatives of the coefficients $a_0, \cdots, a_m$ of a given order, parameters containing derivatives of a higher order, is the process of covariant differentiation.* It is one object of this paper to make this extension, in general, by expressing each and every derivative

$$\frac{\partial^r a_t}{\partial x_1^{r-s} \partial x_2^s} \quad (t = 0, \cdots, m), \tag{18}$$

as a linear combination of a certain finite number of concomitants constructible from the formulas (17).

V. LINEAR DIFFERENTIAL COVARIANTS. A series of linear covariantive parameters may be obtained by interpreting the invariant relation

$$df'_{\pm 1} = \rho_{\pm 1}^{-1} df_{\pm 1}, \tag{19}$$

as a case of (17). The first of these formulas now becomes

$$\begin{aligned}&[(\beta_0, a_1)' + (\beta_0, \beta_1)' \pm (\beta_0, \Delta)'] dy_1 + [(\beta_1, a_1)' \pm (\Delta, a_1)'] dy_2 \\ &= \rho_{\pm 1}^{-1} D\{[(\beta_0, a_1) + (\beta_0, \beta_1) \pm (\beta_0, \Delta)] dx_1 + [(\beta_1, a_1) \pm (\Delta, a_1)] dx_2\},\end{aligned} \tag{20}$$

where primed jacobian symbols indicate derivatives with reference to y_1, y_2, *i. e.*

$$(\beta_0, a_1)' = \frac{\partial \beta_0}{\partial y_1} \frac{\partial a_1}{\partial y_2} - \frac{\partial \beta_0}{\partial y_2} \frac{\partial a_1}{\partial y_1}. \tag{21}$$

The covariants (20), viz.,

$$q'_{+1} = \rho_{+1}^{-1} D q_{+1}, \quad q'_{-1} = \rho_{-1}^{-1} D q_{-1}, \tag{22}$$

are functionally independent of df_{+1}, df_{-1} under the theory that the derivative of an arbitrary function is an arbitrary function. It is legitimate to equate q_{+1} to exact differentials of independent functions, in fact, with

$$\begin{aligned} d\zeta_{+1}^{(1)} &= \frac{\partial \zeta_{+1}^{(1)}}{\partial x_1} dx_1 + \frac{\partial \zeta_{+1}^{(1)}}{\partial x_2} dx_2 \equiv q_{+1}, \\ d\zeta_{-1}^{(1)} &= \frac{\partial \zeta_{-1}^{(1)}}{\partial x_1} dx_1 + \frac{\partial \zeta_{-1}^{(1)}}{\partial x_2} dx_2 \equiv q_{-1}, \end{aligned} \tag{23}$$

only the condition

* Wright, *Invariants of Quadratic Differential Forms*, p. 20 (Historical), p. 11.

$$(24)\qquad \lambda = (\Delta, \beta_0)(\beta_1, \alpha_1) + (\Delta, \alpha_1)(\beta_0, \beta_1) + (\Delta, \alpha_1)(\beta_0, \alpha_1) \neq 0,$$

is implied and this is evidently satisfied since the expression is not identically zero and the functions are arbitrary. In deriving (20) we took $u_1 = \rho_{\pm 1}^{-1}$ so that the condition $(u_1, R) = 0$ is identically satisfied.

If this process is repeated by employing (22) as a case of the relation $I' = RI$ and taking $u_1 = \rho_{\pm 1}^{-1}$, thus satisfying the equation $(u_1, R) = 0$ identically, we obtain

$$\begin{aligned}(25)\quad &[((\beta_0, \alpha_1), \beta_1)' + ((\beta_0, \beta_1), \beta_1)' + ((\beta_0, \alpha_1), \alpha_1)' + ((\beta_0, \beta_1), \alpha_1)' \\ &\quad + ((\beta_0, \Delta), \Delta)' \pm ((\beta_0, \Delta), \beta_1)' \pm ((\beta_0, \alpha_1), \Delta)' \\ &\quad \pm ((\beta_0, \beta_1), \Delta)' \pm ((\beta_0, \Delta), \alpha_1)']\, dy_1 \\ &+ [((\beta_1, \alpha_1), \beta_1)' + ((\beta_1, \alpha_1), \alpha_1)' + ((\Delta, \alpha_1), \Delta)' \pm ((\Delta, \alpha_1), \beta_1)' \\ &\quad \pm ((\Delta, \alpha_1), \alpha_1)' \pm ((\beta_1, \alpha_1), \Delta)']\, dy_2 \\ &= \rho_{\pm 1}^{-1} D\{[((\beta_0, \alpha_1), \beta_1) + ((\beta_0, \beta_1), \beta_1) + ((\beta_0, \alpha_1), \alpha_1) + ((\beta_0, \beta_1), \alpha_1) \\ &\quad + ((\beta_0, \Delta), \Delta) \pm ((\beta_0, \Delta), \beta_1) \pm ((\beta_0, \Delta), \alpha_1) \pm ((\beta_0, \beta_1), \Delta) \\ &\quad \pm ((\beta_0, \alpha_1), \Delta)]\, dx_1 \\ &+ [((\beta_1, \alpha_1), \beta_1) + ((\beta_1, \alpha_1), \alpha_1) + ((\Delta, \alpha_1), \Delta) \pm ((\Delta, \alpha_1), \beta_1) \\ &\quad \pm ((\Delta, \alpha_1), \alpha_1) \pm ((\beta_1, \alpha_1), \Delta)]\, dx_2\},\end{aligned}$$

that is,

$$(26)\qquad d\zeta_{\pm 1}^{(2)\prime} = \rho_{\pm 1}^{-1} D d\zeta_{\pm 1}^{(2)}.$$

The number of possible repetitions and in consequence the number of conjugate pairs of linear differential covariants that can be found is unlimited and it is important to prove the theorem: *The first n of the linear covariants*

$$(27)\qquad df_{+1},\ d\zeta_{+1}^{(1)},\ d\zeta_{+1}^{(2)}, \cdots,\ d\zeta_{+1}^{(n)}, \cdots,$$

are linearly independent in $R(1, T^{(n)}, \Delta)$.

A linear relation between the covariants would imply such a relation to connect the n coefficients of dx_1 in the respective quantics df_{+1}, $d\zeta_{+1}^{(i)}$ $(i = 1, \cdots, n-1)$ and these coefficients, written in their simplest forms, are

$$(28)\qquad \beta_0,\ (\beta_0, \rho_{+1}^{-1}),\ ((\beta_0, \rho_{+1}^{-1}), \rho_{+1}^{-1}), \cdots,\ (\cdots(\beta_0, \overbrace{\rho_{+1}^{-1}), \cdots, \rho_{+1}^{-1}}^{n-1}).$$

But, with δ arbitrary, there exists no formal identity of the type

$$(29)\qquad d_0\beta_0 + d_1(\beta_0, \delta) + d_2((\beta_0, \delta), \delta) + \cdots + d_{n-1}(\cdots(\beta_0, \overbrace{\delta), \delta, \cdots, \delta}^{n-1}) = 0.$$

VI. INFINITE SETS IN $R(1, T^{(n)}, \Delta)$. Any pair of conjugate covariants

$d\zeta_{\pm 1}^{(e)}$, corresponding to the system (27), may be employed as arguments in an expansion of the ground-form F, the coefficients in such an expansion being differential parameters of a type analogous to ϕ_{m-2i} $(i=0, \cdots, m)$ (Cf. (7)). Let these coefficients be

$$\Psi_{m-2i}^{(e,\,0)}, \quad (i=0, \cdots, m). \tag{30}$$

A survey of their structure is to be obtained from the case $e=1$:

$$\begin{aligned}\Psi_{m-2i}^{(1,\,0)} =&\{[(\Delta, a_1)-(\beta_1, a_1)]f_1+[(\Delta, \beta_0)+(\beta_0, a_1)+(\beta_0, \beta_1)]f_2\}^{m-i}\\ &\times\{[(\Delta, a_1)+(\beta_1, a_1)]f_1+[(\Delta, \beta_0)-(\beta_0, a_1)-(\beta_0, \beta_1)]f_2\}^{i}\\ &\div 2^m\lambda^m, \quad (i=0, \cdots, m).\end{aligned} \tag{31}$$

When $m=2$ we then have

$$\begin{aligned}\Psi_{\pm 2}^{(1,\,0)} =&\{[(\beta_1, a_1)\mp(\Delta, a_1)]^2a_0-2[(\beta_1, a_1)\\ &\mp(\Delta, a_1)][(\beta_0, \beta_1)+(\beta_0, a_1)\pm(\Delta, \beta_0)]a_1\\ &+[(\beta_0, \beta_1)+(\beta_0, a_1)\pm(\Delta, \beta_0)]^2a_2\}\div 4\lambda^2,\end{aligned} \tag{32}$$

$$\begin{aligned}\Psi_{0}^{(1,\,0)} =&\{[(\Delta, a_1)^2-(\beta_1, a_1)^2]a_0+2[(\beta_1, a_1)(\beta_0, \beta_1)+(\beta_0, a_1)(\beta_1, a_1)\\ &+(\Delta, \beta_0)(\Delta, a_1)]a_1+[(\Delta, \beta_0)^2-((\beta_0, a_1)\\ &+(\beta_0, \beta_1))^2]a_2\}\div 4\lambda^2.\end{aligned} \tag{33}$$

In general the formulas of (17) are again applicable and we obtain the sets,

$$\begin{aligned}&\Psi_{m-2i}^{(e,\,0)\prime} = Q^{(i)}\Psi_{m-2i}^{(e,\,0)}, \quad (i=0, \cdots, m;\ e=0, 1, \cdots, n, \cdots),\\ &\Psi_{m-2i}^{(e,\,1)\prime} = (\Psi_{m-2i}^{(e,\,0)\prime}, Q^{(i)\prime}) = Q_1^{(i)}(\Psi_{m-2i}^{(e,\,0)}, Q^{(i)}),\\ &\Psi_{m-2i}^{(e,\,2)\prime} = ((\Psi_{m-2i}^{(e,\,0)\prime}; Q^{(i)\prime}), Q_1^{(i)\prime}) = Q_2^{(i)}((\Psi_{m-2i}^{(e,\,0)}, Q^{(i)}), Q_1^{(i)}),\\ &\cdots\cdots\cdots

$$f = \sum_{i=0}^{m} \binom{m}{i} \Phi^{(1,1)}_{m-2i} df^{m-i}_{+1} df^{i}_{-1}, \tag{37}$$

and it will be plain that the forms $\Phi^{(1,1)}_{m-2i}$ are concomitants of the domain $R(1, T^{(2)}, \Delta)$, or else of $R(1, T^{(1)}, \Delta)$ with u and $\partial u/\partial x_j$ $(j = 1, 2)$ adjoined, which are independent of ϕ_{m-2i} $(i = 0, \cdots, m)$ and whose invariant relations are

$$\Phi^{(1,1)\prime}_{m-2i} = \rho^{m-2i}_{+1} D^{i+1} \Phi^{(1,1)}_{m-2i} \quad (i = 0, \cdots, m). \tag{38}$$

A simple rule for obtaining their explicit forms is to substitute, for $a_0, \cdots, a_m$, in the invariants ϕ_{m-2i}, the respective formulas

$$\frac{\partial u}{\partial x_2}\frac{\partial a_r}{\partial x_1} - \frac{\partial u}{\partial x_1}\frac{\partial a_r}{\partial x_2} \quad (r = 0, \cdots, m). \tag{39}$$

If we apply to the forms (38) the methods of (17) we have the sets

$$\begin{aligned}
&\psi^{(1,2)\prime}_{m-2i} = (\Phi^{(1,1)\prime}_{m-2i}, Q^{(i)\prime}_{1}) = Q^{(i)}_{2}(\Phi^{(1,1)}_{m-2i}, Q^{(i)}_{1}), \ (i = 0, \cdots, m),\\
&\psi^{(1,3)\prime}_{m-2i} = ((\Phi^{(1,1)\prime}_{m-2i}, Q^{(i)\prime}_{1}), Q^{(i)\prime}_{2}) = Q^{(i)}_{3}((\Phi^{(1,1)}_{m-2i}, Q^{(i)}_{1}), Q^{(i)}_{2}),\\
&\qquad \cdots \cdots \cdots

Δ) and N_{jr} are linear combinations, in that domain, of the differential parameters $\Psi_{m-2i}^{(0,\,0)} = \phi_{m-2i}$ $(i = 0, \cdots, m)$.

In general the number of n-th derivatives of functions a_r which are involved in the set of $(n+1)(m+1)$ iterated transvectants

$$\Psi_{m-2i}^{(e,\,n)} \quad (e = 0, \cdots, n;\ i = 0, \cdots, m), \tag{43}$$

is equal to the number of these differential parameters, viz., $(n+1)(m+1)$. Hence, by solving a linear system by Cramer's rule, we can express the n-th derivatives of a_r as linear non-homogeneous functions of iterated jacobians, *e. g.*,

$$\frac{\partial^n a_r}{\partial x_1^{n-s}\partial x_2^{s}} = \sum_{e=0}^{n}\sum_{i=0}^{m} \gamma_{s,\,i}^{(e,\,r)}\Psi_{m-2i}^{(e,\,n)} + N_{sr} \quad (s = 0, \cdots, n;\ r = 0, \cdots, m). \tag{44}$$

In these formulas the coefficients γ are expressions which appertain to $R(1, T^{(n)}, \Delta)$ whereas each function N_{sr} is a function of the parameters $\Psi_{m-2i}^{(e,\,\nu)}$ but contains no iterated transvectant of upper index $\nu > n-1$. The expressions N_{sr} are linear rational integral functions, in $R(1, T^{(n)}, \Delta)$, of their jacobian arguments and of ϕ_{m-2i} $(i = 0, \cdots, m)$.

If we should be concerned with a differential parameter which contains derivatives of an arbitrary function u other than those which are derivatives of $x_j(y_1, y_2)$ $(j = 1, 2)$, we would be able, by similar methods, to express them in terms of jacobians of the types

$$(u, \tau_1),\ ((u, \tau_1), \tau_2),\ \cdots, \tag{45}$$

$\tau_1, \tau_2, \cdots$ being chosen functions of derivatives of $x_j(y_1, y_2)$. Moreover $a_0, \cdots, a_m$ are linear in ϕ_{m-2i}, their expressions being obtainable as the inverse of the relations

$$\phi_{m-2i} = \phi_{m-2i}(a_0, \cdots, a_m). \tag{46}$$

For example, if $m = 2$, we find

$$\begin{aligned} a_0 &= 4\beta_0^2(\phi_2 + \phi_0 + \phi_{-2}), \\ a_1 &= 2\beta_0[(\gamma_1 + \Delta)\phi_2 + \gamma_1\phi_0 + (\gamma_1 - \Delta)\phi_{-2}], \\ a_2 &= (\gamma_1 + \Delta)^2\phi_2 - 4a_2\beta_0\phi_0 + (\gamma_1 - \Delta)^2\phi_{-2}. \end{aligned} \tag{47}$$

A summary of the conclusions numbered (42) to (47) gives the following:

THEOREM. *A fundamental system, in the domain $R(1, T^{(n)}, \Delta)$, of invariantive and covariantive differential parameters,*

$$I\left(a_r; \frac{\partial^n a_r}{\partial x_1^{n-s}\partial x_2^{s}}; \frac{\partial^t u}{\partial x_1^{t-v}\partial x_2^{v}}\ ; dx_1, dx_2\right), \tag{48}$$

of the n-th order, of an arbitrary differential quantic $F = (df)^m = (d\phi)^m$, *consists, in addition to arbitrary invariants of the types* (u, τ_1), $((u, \tau_1), \tau_2)$, $\cdots$, *of*

$$\tfrac{1}{2}(n+1)(n+2)(m+1)+2 \tag{49}$$

parameters which may be selected in sets as follows:

$$df_{\pm 1}\ ,\ \Psi_{m-2i}^{(e,\,\nu)} \quad (i=0,\cdots,m;\ \nu=0,\cdots,n;\ e=0,\cdots,\nu). \tag{50}$$

Parameters which involve arbitrary functions u. The known linear relations,

$$D\frac{\partial}{\partial x_2} = a_1\frac{\partial}{\partial y_2} + a_2\left(-\frac{\partial}{\partial y_1}\right), D\left(-\frac{\partial}{\partial x_1}\right) = \beta_0\frac{\partial}{\partial y_2} + \beta_1\left(-\frac{\partial}{\partial y_1}\right),$$

establish the congrediency, for the transformation of homogenous covariants, of the sets dx_1, dx_2; $\partial/\partial x_2$, $-\partial/\partial x_1$. A differential parameter of the type (48), which contains derivatives of u, may be regarded, therefore, as a part of a concomitant that has emanated from a purely covariantive concomitant, $J(a_s, \partial^n a_s/\partial x_1^{n-v}\partial x_2^v;\ dx_1,\ dx_2)$, by the polar operation

$$O = dy_1\frac{\partial}{\partial(dx_1)} + dy_2\frac{\partial}{\partial(dx_2)}\Bigg]\begin{matrix} dy_1 = \partial/\partial x_2. \\ dy_2 = -\partial/\partial x_1 \end{matrix} \tag{51}$$

Thus the theory of parameters which contain u is represented by that of concomitants which do not contain u or its derivatives, in combination with the theory of the invariantive process of polarization. We confine attention, therefore, to parameters which involve no arbitrary functions u.

IX. COMPLETE SYSTEMS IN $R\ (1, T^{(n)}, 0)$. A differential parameter of the field $R\ (1, T^{(n)}, 0)$ is a compound of binomial summands, of the general form

$$I_{\pm 1} = \Sigma(R_{+1}P_{+1} \pm R_{-1}P_{-1}), \tag{52}$$

where P_{+1} represents a monomial product of invariant elements and P_{-1} is the conjugate of that product with reference to the double sign of the radical Δ. Hence,

$$P_{+1} = \prod_{i=0}^{m}\prod_{e=0}^{\nu}\prod_{\nu=0}^{n} \Psi_{m-2i}^{(e,\,\nu)x_i e_\nu} df_{+1}^{\sigma_1} df_{-1}^{\sigma_2}, \tag{53}$$

$$P_{-1} = \prod_{i=0}^{m}\prod_{e=0}^{\nu}\prod_{\nu=0}^{n} \Psi_{-(m-2i)}^{(e,\,\nu)x_i e_\nu} df_{-1}^{\sigma_1} df_{+1}^{\sigma_2},$$

while $R_{\pm 1}$ are conjugate polynomials, rational in the derivatives of the functions $x_j(y_1, y_2)\ (j=1,2)$, and in Δ. An important fact in connection with a differential parameter of any domain, when expressed as a polynomial in $\Psi_{m-2i}^{(e,\,\nu)}$, $df_{\pm 1}$, is that of term-wise invariance. Each term of $I_{\pm 1}$ is invariantive

and therefore they must have common indices with reference to the powers of ρ_{+1}, D, which form the multiplier in an invariant relation. We find, from (3), (6), (9);

$$P'_{+1} = \rho_{+1}^{h} D^{k_1} P_{+1}\ ,\ P'_{-1} = \rho_{+1}^{-h} D^{k_2} P_{-1}\ , \tag{54}$$

with the following expressions as the explicit values of the exponents:

$$h = \sum_{i=0}^{m} \sum_{e=0}^{\nu} \sum_{\nu=0}^{n} (m - 2i) x_{ie\nu} - \sigma_1 + \sigma_2, \tag{55}$$

$$k_1 = \sum_{i=0}^{m} \sum_{e=0}^{\nu} \sum_{\nu=0}^{n} (i + \nu) x_{ie\nu} - \sigma_2, \tag{56}$$

$$k_2 = \sum_{i=0}^{m} \sum_{e=0}^{\nu} \sum_{\nu=0}^{n} (m + \nu - i) x_{ie\nu} - \sigma_1.$$

The variables x_{ie} assume only positive integral values but are unrestricted otherwise.

A necessary and sufficient condition for the term-wise invariance of I is, therefore, that the linear diophantine equation, $h = 0$, should be satisfied. But, according to a lemma due to Hilbert, monomials chosen with their indices satisfying $h = 0$ form a finite system in that they are divisible by at least one of a finite set properly chosen from the infinitude. The set is given, in fact, by the finite set of irreducible solutions, in the variables $x_{ie\nu}$, of $h = 0$. Taking notice of the identities

$$P_{\pm 1} = \tfrac{1}{2}(P_{+1} + P_{-1}) \pm \tfrac{1}{2}(P_{+1} - P_{-1}), \tag{57}$$

we have the following,

THEOREM. *The finite set of irreducible solutions of the linear diophantine equation $h = 0$ gives a finite complete system of invariantive and covariantive, relative and absolute, differential parameters of order n in $R(1, T^{(n)}, 0)$. The fundamental parameters are of the general types*

$$T_{\pm 1} = P_{+1} \pm P_{-1}, \tag{58}$$

where P_{+1} and its conjugate P_{-1} are determined by (conjugate) irreducible solutions.

The concomitants here dealt with have been termed, by the present writer, in another paper, parameters of the extended orthogonal type and one such system,* that of the quadratic, with e restricted to the value zero, has been determined. It is of comparative import to give the explicit expression for

* *Annals of Math.*, loc. cit., p. 27.

one such parameter and we may select, from the before-mentioned system, the covariant

$$r_{+1} = (\phi_2, \rho^2_{+1})\, df^2_{+1} + (\phi_{-2}, \rho^2_{-1})\, df^2_{-1}, \tag{59}$$

of order one, the invariant relation for which is $r'_{+1} = Dr_{+1}$. We employ the notation (AB) for the determinant $A_1B_2 - A_2B_1$, this being different from the notation (A, B) for jacobians, and use the abbreviations shown in the footnote * for certain eight rational expressions in the x_1, x_2 derivatives of the functions a_1, a_2, β_0, β_1. We also designate by θ_1, θ_2 the following linear expressions in the derivatives of the functions a_j $(j = 0, 1, 2)$:

$$\begin{aligned}\theta_1 = \{&(-\gamma_1 G_2 + FH_2)\frac{\partial a_0}{\partial x_1} + (2\beta_0 G_2 - 2\beta_0\gamma_1 H_2)\frac{\partial a_1}{\partial x_1} \\ &+ 2\beta_0{}^2 H_2 \frac{\partial a_2}{\partial x_1} + (\gamma_1 G_1 - FH_1)\frac{\partial a_0}{\partial x_2} + (-2\beta_0 G_1 + 2\beta_0\gamma_1 H_1)\frac{\partial a_1}{\partial x_2} \\ &- 2\beta_0{}^2 H_1 \frac{\partial a_2}{\partial x_2} + \beta_0{}^{-2}\Delta^{-2}[((BG)+(AH))a_0 \\ &+ ((DG)+(CH))a_1 + (EH)a_2]\} \div (2\beta_0\Delta)^2,\end{aligned} \tag{60}$$

$$\begin{aligned}\theta_2 = \{&(FG_2 - \gamma_1 H_2\Delta^2)\frac{\partial a_0}{\partial x_1} + (-2\beta_0\gamma_1 G_2 + 2\beta_0 H_2\Delta^2)\frac{\partial a_1}{\partial x_1} \\ &+ 2\beta_0{}^2 G_2 \frac{\partial a_2}{\partial x_1} - 2\beta_0{}^2 G_1 \frac{\partial a_2}{\partial x_2} - (FG_1 - \gamma_1 H_1\Delta^2)\frac{\partial a_0}{\partial x_2} \\ &-(-2\beta_0\gamma_1 G_1 + 2\beta_0 H_1\Delta^2)\frac{\partial a_1}{\partial x_2} \\ &+ \beta_0{}^{-2}\Delta^{-2}[((AG)+(BH)\Delta^2)a_0 + ((CG)+(DH)\Delta^2)a_1 \\ &+ (EG)a_2]\} \div (2\beta_0\Delta)^2.\end{aligned} \tag{61}$$

The explicit form of the covariantive parameter in question is then

$$\begin{aligned}r_{+1} = {}&2\beta_0{}^2\theta_1 dx_1{}^2 + 2\beta_0(\gamma_1\theta_1 + \theta_2)\, dx_1 dx_2 \\ &+ (\gamma_1{}^2\theta_1 + 2\beta_0 a_2\theta_1 + \gamma_1\theta_2)\, dx_2{}^2.\end{aligned} \tag{62}$$

In Part 2 this concomitant is much simplified due to the hypothesis that $x_1 + ix_2$, $(i = \sqrt{-1})$, is an analytic function.

---

* $A_i = \Delta^2\beta_0{}^2(2\gamma_1\gamma_{1x_i} + 2\beta_0 a_{1x_i} + 2a_2\beta_{0x_i}) - F(2\Delta^2\beta_0\beta_{0x_i} + 2\beta_0{}^2\gamma_1\gamma_{1x_i} + 4\beta_0{}^3 a_{2x_i} + 4\beta_0{}^3 a_2\beta_{0x_i})$,
$B_i = -\Delta^2\beta_0{}^3\gamma_{1x_i} + 2\beta_0\gamma_1\Delta^2\beta_{0x_i} + \beta_0{}^3\gamma_1{}^3\gamma_{1x_i} + 2\beta_0{}^3\gamma_1 a_{1x_i} + 2\beta_0{}^3 a_1\gamma_1\beta_{0x_i}$,
$C_i = -2\beta_0{}^3\Delta^2\gamma_{1x_i} + 2\beta_0{}^3\gamma_1\Delta^2\beta_{0x_i} + 4\beta_0{}^3\gamma_1{}^2\gamma_{1x_i} + 8\beta_0{}^4\gamma_1 a_{1x_i} + 8\beta_0{}^3\gamma_1 a_2\beta_{0x_i}$,
$D_i = -2\beta_0{}^3\Delta^2\beta_{0x_i} - 2\beta_0{}^3\gamma_1\gamma_{1x_i} - 4\beta_0{}^4 a_{1x_i} - 4\beta_0{}^3 a_2\beta_{0x_i}$,
$E_i = -4\beta_0{}^4\gamma_1\gamma_{1x_i} - 8\beta_0{}^5 a_{1x_i} - 8\beta_0{}^4 a_2\beta_{0x_i}$; $F = \gamma_1{}^2 + 2a_2\beta_0$,
$G_i = \Delta^2(a_{1x_i} + \beta_{1x_i}) + (a_1 + \beta_1)(\gamma_1\gamma_{1x_i} + 2\beta_0 a_{2x_i} + 2a_2\beta_{0x_i})$,
$H_i = 2a_1 a_{1x_i} + 2\beta_0 a_{2x_i} + 2a_2\beta_{0x_i} + 2\beta_1\beta_{1x_i}$; $(i = 1, 2;\ a_{1x_i} = \partial a_1/\partial x_i)$.

X. COMPLETE SYSTEM OF ORDER 2, IN R $(1, T^{(2)}, 0)$, OF THE CUBIC. When $m = 3$, $n = 2$, we are concerned with a fundamental system of parameters of order 2 of

$$F = a_0 dx_1^3 + 3a_1 dx_1^2 dx_2 + 3a_2 dx_1 dx_2^2 + a_3 dx_2^3.$$

The equation $h = 0$, from (55), becomes

$$3x + y + \sigma_2 = 3z + w + \sigma_1; \tag{63}$$

$$\begin{aligned} x &= x_{000} + x_{001} + x_{011} + x_{002} + x_{012} + x_{022}, \\ y &= x_{100} + x_{101} + x_{111} + x_{102} + x_{112} + x_{122}, \\ w &= x_{200} + x_{201} + x_{211} + x_{202} + x_{212} + x_{222}, \\ z &= x_{300} + x_{301} + x_{311} + x_{302} + x_{312} + x_{322}. \end{aligned} \tag{64}$$

In solving (63) we observe that a set $(x, y, z, w, \sigma_1, \sigma_2)$ which is reducible corresponds only to a reducible set $(x_{ijk}\sigma_l)$; therefore, the irreducible solutions of this equation in $x, y, z, w, \sigma_1, \sigma_2$ are written* first. It is then very simple to write all irreducible sets $(x_{ijk}\sigma_l)$ which satisfy (63). It is found that their number is 1153 and in the tabulation below we represent a solution by $S = (x_{000}, x_{001}, \cdots, x_{322}; \sigma_1, \sigma_2)$, the letters x_{ijk} being arranged in ascending order according to the rule that if x_{ijk}, x_{lmn} are any two variables and the first one of the differences $k - n$, $j - m$, $i - l$, which is not zero, is positive, then x_{lmn} precedes x_{ijk}. An irreducible solution is (10000000000000000210000;00), but to secure a compact notation we represent, by a number in parenthesis, the number of zero elements between two elements which are not zero. Accordingly the specimen solution above is [1(17)21(4)00].

According to the theorem in Paragraph IX we expect the totality of solutions to occur in conjugate pairs, each pair appertaining to an irreducible form $T_{\perp 1}$ which it determines. This is true of all solutions excepting thirteen that are self-conjugate. The latter give products $P_{\perp 1}$ which are unaltered by the changes of sign of all subscripts and, in this case, the concomitant $P_{\perp 1}$ belongs to $R(1, T^{(2)}, 0)$. We give only one solution of each conjugate pair and thus simplify, without loss of clearness, the tabular representation of the fundamental system of F in $R(1, T^{(2)}, 0)$. When the systems of solutions are written *in extenso* it is found that certain sets can be comprised in one formula, involving α, β, varying independently from 0 to 5. (*$\gamma = 0, \cdots, 4$).

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* *Annals of Math.*, Vol. 20 (1918), p. 131.

Invariantive parameters.

$[(a)1(17-a+\beta)1(5-\beta)00]$, $[(6+a)1(5-a+\beta)1(11-\beta)00]$,
$[(a)1(11-a+\beta)3(11-\beta)00]$, $[(a)1(11-a)2(\gamma)1(10-\gamma)00]$,* $[(a)1(11-a)12(10)00]$,
$[(a)1(12-a)21(9)00]$, $[(a)1(12-a)2(1)1(8)00]$, $[(a)1(12-a)2(2)1(7)00]$,
$[(a)1(12-a)2(3)1(6)00]$, $[(a)1(11-a)1(1)2(9)00]$, $[(a)1(12-a)12(9)00]$,
$[(a)1(13-a)21(8)00]$, $[(a)1(13-a)2(1)1(7)00]$, $[(a)1(13-a)2(2)1(6)00]$,
$[(a)1(11-a)1(2)2(8)00]$, $[(a)1(12-a)1(1)2(8)00]$, $[(a)1(13-a)12(8)00]$,
$[(a)1(14-a)21(7)00]$, $[(a)1(14-a)2(1)1(6)00]$, $[(a)1(11-a)1(3)2(7)00]$,
$[(a)1(12-a)1(2)2(7)00]$, $[(a)1(13-a)1(1)2(7)00]$, $[(a)1(14-a)12(7)00]$,
$[(a)1(15-a)21(6)00]$, $[(a)1(11-a)1(4)2(6)00]$, $[(a)1(12-a)1(3)2(6)00]$,
$[(a)1(13-a)1(2)2(6)00]$, $[(a)1(14-a)1(1)2(6)00]$, $[(a)1(15-a)12(6)00]$,
$[(a)1(11-a)111(9)00]$, $[(a)1(11-a)11(1)1(8)00]$, $[(a)1(11-a)11(2)1(7)00]$,
$[(a)1(11-a)11(3)1(6)00]$, $[(a)1(11-a)1(1)11(8)00]$, $[(a)1(11-a)1(1)1(1)1(7)00]$,
$[(a)1(11-a)1(1)1(2)1(6)00]$, $[(a)1(11-a)1(2)11(7)00]$, $[(a)1(11-a)1(2)1(1)1(6)00]$,
$[(a)1(11-a)1(3)11(6)00]$, $[(a)1(12-a)111(8)00]$, $[(a)1(12-a)11(1)1(7)00]$,
$[(a)1(12-a)11(2)1(6)00]$, $[(a)1(12-a)1(1)11(7)00]$, $[(a)1(12-a)1(1)1(1)1(6)00]$,
$[(a)1(12-a)1(2)11(6)00]$, $[(a)1(13-a)111(7)00]$, $[(a)1(13-a)11(1)1(6)00]$,
$[(a)1(13-a)1(1)11(6)00]$, $[(a)1(14-a)111(6)00]$.

Covariantive parameters.

$[(6+a)1(17-a)10]$, $[(a)1(23-a)30]$, $[(a)1(11-a+\beta)1(11-\beta)20]$,
$[(a)1(11-a+\beta)2(11-\beta)10]$, $[(a)1(11-a)11(10)10]$, $[(a)1(11-a)1(1)1(9)10]$,
$[(a)1(11-a)1(2)1(8)10]$, $[(a)1(11-a)1(3)1(7)10]$, $[(a)1(11-a)1(4)1(6)10]$,
$[(a)1(12-a)11(9)10]$, $[(a)1(12-a)1(1)1(8)10]$, $[(a)1(12-a)1(2)1(7)10]$,
$[(a)1(12-a)1(3)1(6)10]$, $[(a)1(13-a)11(8)10]$, $[(a)1(13-a)1(1)1(7)10]$,
$[(a)1(13-a)1(2)1(6)10]$, $[(a)1(14-a)11(7)10]$, $[(a)1(14-a)1(1)1(6)10]$,
$[(24)11]$, $[(a)1(15-a)11(6)10]$.

The general method of writing the irreducible parameters $T_{\pm 1}$ (cf. (34), (52)), from their symbols, can be made plain by an example. The product corresponding to the solution

$$[(2)1(12)2(8)10] = (00100000000000020000000010),$$

is $\Psi_{+8}^{(11)}\Psi_{-1}^{(02)^2}df_{+1}$ and its conjugate is $\Psi_{-8}^{(11)}\Psi_{+1}^{(02)^2}df_{-1}$.

Hence we have

$$(65) \qquad F_{\pm 1} = (\Psi_3^{(10)}, Q^{(0)})((\phi_{-1}, Q^{(2)}), Q_1^{(2)})^2 df_{+1} \pm (\Psi_{-3}^{(10)}, Q^{(3)})((\phi_{+1}, Q^{(1)}), Q_1^{(1)})^2 df_{-1}.$$

XI. THE DOMAIN R (1, 0, 0). The differential parameters involved in relativity and all parameters, in fact, treated in researches prior to the present writer's paper of 1921, are explicitly free from the functions and the derivatives thereof which are contained in the transformations. These may be said to appertain to the numerical domain $R(1, 0, 0)$ but the present theory shows that they are, notwithstanding, rational, integral forms in the elements (50) (cf. (65) of such a type that all expressions in a_1, a_2, β_0, β_1 and their derivatives factor out and disappear by cancellation. The problem of the complete system is now a more difficult problem. But we derive an important truth from the famous lemma of Hilbert to the effect that a quantic, F, belonging to a system constructed according to any law, is a linear combination of a finite set from the system,

$$F = Z_1F_1 + Z_2F_2 + \cdots + Z_sF_s, \tag{66}$$

where the Z_i, although homogeneous polynomials in the same finite number of variables, do not necessarily belong to the original system.

We may take the variables to be the set (50) and assume for the law of the system merely the hypothesis that a form F shall be of such structure that it simplifies into the domain $R(1, 0, 0)$. This gives an infinite system, that is, the totality of parameters of the domain $R(1, 0, 0)$: Hence every such concomitant F is linear in a finite set, as in (66), where $F_1, \cdots, F_s$ also belong to the domain $R(1, 0, 0)$. It is also plain that Z_i appertain at least to $R(1, T^{(n)}, 0)$ if not to $R(1, 0, 0)$, and hence are reducible in terms of a finite set such as was derived explicitly in Paragraph X. A proof that Z_i can be chosen so that they, too, belong to $R(1, 0, 0)$, is to be desired but it requires an undiscovered algorism analogous to the one employed by Hilbert in the case of algebraic concomitants. This question is waived. We pass to the second division of the paper where some parameters which belong nominally to $R(1, T^{(n)}, 0)$ simplify, on account of intrinsic properties, to the domain $R(1, 0, 0)$.

* * *

PART 2.

XII. TRANSFORMATIONS WITH CONDITIONS. We may assume, here, conditions which $x_1(y_1, y_2)$, $x_2(y_1, y_2)$ may satisfy without ceasing to be arbitrary; for instance, we first take the relations below, by postulate,

$$\frac{\partial x_1}{\partial y_2} = -\frac{\partial x_2}{\partial y_1}, \quad \frac{\partial x_1}{\partial y_1} = 2\epsilon\frac{\partial x_2}{\partial y_1} + \frac{\partial x_2}{\partial y_2}, \tag{67}$$

the quantity ϵ being a fixed complex number while x_1, x_2 are real functions.

If $\epsilon = 0$ this hypothesis can all be included in the statement that $x_1 + ix_2$ $(i = \sqrt{-1})$ is analytic. The transformations upon the differentials become

$$(68)\quad T_1: \begin{cases} dx_1 = (2\epsilon \dfrac{\partial x_2}{\partial y_1} + \dfrac{\partial x_2}{\partial y_2})\, dy_1 - \dfrac{\partial x_2}{\partial y_1}\, dy_2, \\ dx_2 = \dfrac{\partial x_2}{\partial y_1}\, dy_1 + \dfrac{\partial x_2}{\partial y_2}\, dy_2, \end{cases}$$

and the poles $r_{\pm 1}$ of T_1 are fixed: $r_{\pm 1} = \epsilon \pm \sqrt{\epsilon^2 - 1} \equiv \epsilon \pm \delta$. The linear covariants,

$$(69)\qquad df_{\pm 1} = dx_1 + (-\epsilon \pm \sqrt{\epsilon^2 - 1})\, dx_2,$$

are universal, for an arbitrary $x_2(y_1, y_2)$, and in the invariant relations

$$df'_{\pm 1} = \rho_{\pm 1}^{-1}\, df_{\pm 1},$$

we have

$$(70)\qquad \rho_{\pm 1} = (\epsilon \pm \delta)\, \frac{\partial x_2}{\partial y_1} + \frac{\partial x_2}{\partial y_2}.$$

The determinant of T_1 is

$$(71)\qquad d = \rho_{+1}\rho_{-1} = \left(\frac{\partial x_2}{\partial y_1}\right)^2 + 2\epsilon\, \frac{\partial x_2}{\partial y_1}\, \frac{\partial x_2}{\partial y_2} + \left(\frac{\partial x_2}{\partial y_2}\right)^2, \quad (= \Delta_1 x_2).^*$$

The covariant

$$(72)\qquad g = df_{+1} df_{-1} = dx_1^2 - 2\epsilon dx_1 dx_2 + dx_2^2, \quad (g' = d^{-1} g),$$

has constant coefficients and accordingly, in developing the present theory by analogy with that of Part 1, all expressions which are derived under the assumption that the coefficients in $df_{\pm 1}$ are differentiable functions vanish. This removes from the theory all linear covariants of the sequence (27) excepting the first one and hence the method, in Part 1, of extending the theory to parameters of order $n > 0$ now fails. The concomitants ϕ_{m-2i} themselves, in fact, are linear forms in $a_0, \cdots, a_m$ with constant coefficients;

$$(73)\qquad \phi_{m-2i} = [(-\epsilon - \delta) f_1 - f_2]^{m-i} [(\epsilon - \delta) f_1 + f_2]^i / (-2\delta)^m,$$

but, with the special values indicated, we still have

$$(74)\qquad \phi'_{m-2i} = \rho_{+1}^{m-2i}\, d^i \phi_{m-2i} \quad (i = 0, \cdots, m),$$

and the suite (9) still exists. The four domains of rationality can be represented, in relation to their defining numbers, as $R(\epsilon, T^{(n)}, \delta)$, $R(\epsilon, T^{(n)})$, $R(\epsilon)$, $R(1)$.

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* Bianchi, *Vorlesungen ueber Differentialgeometrie*, (1899), pp. 41, 47.

It is desirable to notice the reduced form of one parameter derived from (27), which we choose as r_{+1} of (62), Paragraph IX. For brevity we write

$$
\begin{aligned}
6\epsilon^2\delta^2 \qquad \zeta_1 = {} & (\epsilon^4 + + \delta^4)\beta_0(a_0, \beta_0) + (\epsilon^3 + 3\epsilon\delta^2)(a_0, \beta_0\beta_1) \\
& + (\epsilon^2 + \delta^2)\beta_1(a_0, \beta_1) + 2(\epsilon^3 + 3\epsilon\delta^2)\beta_0(a_1, \beta_0) \\
(75) \qquad & + 2(\epsilon^2 + \delta^2)(a_1, \beta_0\beta_1) + 2\epsilon\beta_1(a_1, \beta_1) + (\epsilon^2 + \delta^2)\beta_0(a_2, \beta_0) \\
& + \epsilon(a_2, \beta_0\beta_1) \qquad + \beta_1(a_2, \beta_1), \\
\zeta_2 = {} & (4\epsilon^3 + 4\epsilon\delta^2)\beta_0(a_0, \beta_0) + (3\epsilon^2 + \delta^2)(a_0, \beta_0\beta_1) + 2\epsilon\beta_1(a_0, \beta_1) \\
& + (6\epsilon^2 + 2\delta^2)\beta_0(a_1, \beta_0) + 4\epsilon(a_1, \beta_0\beta_1) + 2\beta_1(a_1, \beta_1) \\
& + 2\epsilon\beta_0(a_2, \beta_0) \qquad + (a_2, \beta_0\beta_1);
\end{aligned}
$$

then,

$$\delta^2 r_{+1} = \zeta_1 dx_1^2 - 2(\epsilon\zeta_1 - \delta^2\zeta_2)dx_1dx_2 + (\epsilon^2\zeta_1 + \delta^2\zeta_1 - 2\epsilon\delta^2\zeta_2)dx_2^2.$$

XIII. THE PROBLEM OF COMPLETE SYSTEMS OF ORDER n. The evanescence of the differentials in (27) when T becomes T_1 is restituted by connecting the present theory with that of Paragraph VII. The transformations and invariants are no longer restricted in the sense of (12). The systems are invariantive under all substitutions which satisfy, in addition to the relations (12), the partial differential equations (67).

Consider concomitants of the first order. The argument in VIII, applied to the inverses of the relations

$$(76) \qquad \Psi^{(0,1)}_{m-2i} = \Psi^{(0,1)}_{m-2i}\left(\frac{\partial a_0}{\partial x_1}, \cdots, \frac{\partial a_m}{\partial x_2}\right), \quad \Phi^{(1,1)}_{m-2i} = \Phi^{(1,1)}_{m-2i}\left(\frac{\partial a_0}{\partial x_1}, \cdots \frac{\partial a_m}{\partial x_2}\right),$$

shows that these $2(m+1)$ parameters, with the set ϕ_{m-2i}, $df_{\pm 1}$, form a complete system in $R(\epsilon, T^{(1)}, \delta)$ and for the domain $R(\epsilon, T^{(1)})$ we have only to form the conjugate products corresponding to (53) $(n=1)$ and to satisfy the following equation by irreducible integral sets:

$$(77) \qquad \sum_{i=0}^{m}(m-2i)x_{i00} + \sum_{i=0}^{m}\sum_{e=0}^{1}(m-2i)x_{ie_1} - \sigma_1 + \sigma_2 = 0.$$

For the higher orders we can employ, in the same way, the iterations* $((F, u), v), \cdots ; u, v, w, \cdots$ being any set of functions which satisfy the relations of absolute invariancy $(u' = u, v' = v, \cdots)$ and functional dependence (cf. (17)). The parameters of order zero all appertain to one of the domains $R(1, \epsilon, \delta)$, $R(1, \epsilon)$, $R(1)$, that is, they are free from the functions of the transformations.

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* Maschke, *Transactions Amer. Math. Soc.*, Vol. 1 (1900), p. 199.

XIV. INVARIANTS OF ANALYTIC TRANSFORMATIONS. The emendation (cf. XIII) of the methods of formation of complete systems developed in Part 1, which were necessary in order to obtain definitive construction methods for systems when $x_1 + ix_2$ is assumed to be an analytic function of $z = y_1 + iy_2$, is but typical of what must be done whenever the transformations T are made particular by any hypothesis. For the invariants are then more numerous and additions to the methods which suffice for T are required, or else new methods must be developed. Special transformations of note to which these remarks apply are obtained by assuming that x_1, x_2, are analytic functions of z.

If both x_1 and x_2 are taken to be such functions, then dx_1 and dx_2 prove to be linearly dependent. Let $x_1 = P + iQ$, where $P(y_1, y_2)$, $Q(y_1, y_2)$ are real, x_1 analytic and $x_2(y_1, y_2)$, $(= R)$, real. Then T reduces to

$$(78) \qquad T' : dx_1 = \frac{\partial x_1}{\partial z}(dy_1 + i dy_2),\ dx_2 = R_{y_1} dy_1 + R_{y_2} dy_2;$$

$$D = (P_{y_1} - iP_{y_2})(R_{y_2} - iR_{y_1}).$$

The methods of Part 1 then give important sets of parameters but completeness cannot be assumed.

THE UNIVERSITY OF PENNSYLVANIA.

On the Isodyadic Septimic Equations.

By J. C. Glashan.

To determine whether a given septimic equation is isodyadic and, if it be so, to solve it.

Transform the given equation to the form

$$x^7 - 7p(3x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x) + pa_5 = 0 \qquad (1)$$

Let $x = \omega^n y_1 + \omega^{2n} y_2 + \omega^{3n} y_3 + \omega^{4n} y_4 + \omega^{5n} y_5 + \omega^{6n} y_6$

$$n = 0, 1, 2, 3, 4, 5, 6$$

$$(\omega^7 - 1)/(\omega - 1) = 0$$

On eliminating ω we obtain

$$\begin{aligned}
&x^7 - 7(y_1y_6 + y_2y_5 + y_3y_4)x^5 - 7\{y_1^2y_5 + y_2^2y_3 + y_3^3y_1 + y_4^2y_6 \\
&\qquad + y_5^2y_4 + y_6^2y_2 + 2(y_1y_2y_4 + y_3y_5y_6)\}x^4 \\
&-7\{y_1^3y_4 + y_2^3y_1 + y_3^3y_5 + y_4^3y_2 + y_5^3y_6 + y_6^3y_3 \\
&\quad + 3(y_1^2y_2y_3 + y_2^2y_4y_6 + y_3^2y_2y_6 + y_4^2y_1y_5 + y_5^2y_1y_3 + y_6^2y_4y_5) \\
&\quad - 2(y_1^2y_6^2 + y_2^2y_5^2 + y_3^2y_4^2) - (y_1y_2y_5y_6 + y_2y_3y_4y_5 + y_1y_3y_4y_6)\}x^3 \\
&-7\{y_1^4y_3 + y_2^4y_6 + y_3^4y_2 + y_4^4y_5 + y_5^4y_1 + y_6^4y_4 \\
&\quad + 2(y_1^3y_2^2 + y_2^3y_4^2 + y_3^3y_6^2 + y_4^3y_1^2 + y_5^3y_3^2 + y_6^3y_5^2) \\
&\quad - 3(y_1^3y_5y_6 + y_2^3y_3y_5 + y_3^3y_1y_4 + y_4^3y_3y_6 + y_5^3y_2y_4 + y_6^3y_1y_2) \\
&\quad - (y_1^2y_3^2y_6 + y_2^2y_6^2y_5 + y_3^2y_2^2y_4 + y_4^2y_5^2y_3 \\
&\qquad + y_5^2y_1^2y_2 + y_6^2y_4^2y_1) \\
&\quad - 2(y_1^2y_2y_4y_6 + y_2^2y_1y_4y_5 + y_3^2y_4y_5y_6 + y_4^2y_1y_2y_3 \\
&\qquad + y_5^2y_2y_3y_6 + y_6^2y_1y_3y_5) \\
&\quad + 5(y_1^2y_3y_4y_5 + y_2^2y_1y_3y_6 + y_3^2y_1y_2y_5 + y_4^2y_2y_5y_6 \\
&\qquad + y_5^2y_1y_4y_6 + y_6^2y_2y_3y_4)\}x^2 \\
&-7\{y_1^5y_2 + y_2^5y_4 + y_3^5y_6 + y_4^5y_1 + y_5^5y_3 + y_6^5y_5 \\
&\quad - (y_1^4y_5^2 + y_2^4y_3^2 + y_3^4y_1^2 + y_4^4y_6^2 + y_5^4y_4^2 + y_6^4y_2^2) \\
&\quad - 2(y_1^4y_4y_6 + y_2^4y_1y_5 + y_3^4y_4y_5 + y_4^4y_2y_3 + y_5^4y_2y_6 \\
&\qquad + y_6^4y_1y_3) + y_1^3y_6^3 + y_2^3y_5^3 + y_3^3y_4^3 \\
&\quad + 3(y_1^3y_3^2y_5 + y_2^3y_6^2y_3 + y_3^3y_2^2y_1 + y_4^3y_5^2y_6 + y_5^3y_1^2y_4 \\
&\qquad + y_6^3y_4^2y_2 + y_1^3y_4^2y_3 + y_2^3y_1^2y_6 + y_3^3y_5^2y_2 \\
&\qquad + y_4^3y_2^2y_5 + y_5^3y_6^2y_1 + y_6^3y_3^2y_4) \\
&\quad - (y_1^3y_2y_3y_6 + y_2^3y_4y_5y_6 + y_3^3y_2y_4y_6 + y_4^3y_1y_3y_5 + y_5^3y_1y_2y_3 \\
&\qquad + y_6^3y_1y_4y_5 + y_1^3y_2y_4y_5 + y_2^3y_1y_3y_4 + y_3^3y_1y_5y_6 \\
&\qquad + y_4^3y_1y_2y_6 + y_5^3y_3y_4y_6 + y_6^3y_2y_3y_5)
\end{aligned}$$

$$
\begin{aligned}
&+2(y_1^2y_2^2y_3y_5+y_2^2y_4^2y_3y_6+y_3^2y_6^2y_1y_2+y_4^2y_1^2y_5y_6\\
&\quad+y_5^2y_3^2y_1y_4+y_6^2y_5^2y_2y_4)\\
&-5(y_1^2y_3^2y_2y_4+y_2^2y_6^2y_1y_4+y_3^2y_2^2y_5y_6+y_4^2y_5^2y_1y_2\\
&\quad+y_5^2y_1^2y_3y_6+y_6^2y_4^2y_3y_5)\\
&-5(y_1^2y_6^2y_2y_5+y_2^2y_5^2y_3y_4+y_3^2y_4^2y_1y_6)\\
&\quad+2(y_1^2y_6^2y_3y_4+y_2^2y_5^2y_1y_6+y_3^2y_4^2y_2y_5)\\
&+y_1^2y_2^2y_4^2+y_3^2y_5^2y_6^2+15y_1y_2y_3y_4y_5y_6\}x\\
&-(y_1^7+y_2^7+y_3^7+y_4^7+y_5^7+y_6^7)+7\{y_1^5y_3y_6+y_2^5y_5y_6\\
&\quad+y_3^5y_2y_4+y_4^5y_3y_5+y_5^5y_1y_2+y_6^5y_1y_4+y_1^5y_4y_5\\
&\quad+y_2^5y_1y_3+y_3^5y_1y_5+y_4^5y_2y_6+y_5^5y_4y_6+y_6^5y_2y_3\\
&\quad-(y_1^4y_2^2y_6+y_2^4y_4^2y_5+y_3^4y_6^2y_4+y_4^4y_1^2y_3+y_5^4y_3^2y_2\\
&\qquad+y_6^4y_5^2y_1+y_1^4y_3^2y_4+y_2^4y_6^2y_1+y_3^4y_2^2y_5+y_4^4y_5^2y_2\\
&\qquad+y_5^4y_1^2y_6+y_6^4y_4^2y_3+y_1^4y_4^2y_2+y_2^4y_1^2y_4+y_3^4y_5^2y_6\\
&\qquad+y_4^4y_2^2y_1+y_5^4y_6^2y_3+y_6^4y_3^2y_5+y_1^4y_6^2y_5+y_2^4y_5^2y_3\\
&\qquad+y_3^4y_4^2y_1+y_4^4y_3^2y_6+y_5^4y_2^2y_4+y_6^4y_1^2y_2)\\
&\quad-2(y_1^4y_2y_3y_5+y_2^4y_3y_4y_6+y_3^4y_1y_2y_6+y_4^4y_1y_5y_6+y_5^4y_1y_3y_4\\
&\qquad+y_6^4y_2y_4y_5)\\
&\quad+y_1^3y_2^3y_5+y_2^3y_4^3y_3+y_3^3y_6^3y_1+y_4^3y_1^3y_6+y_5^3y_3^3y_4\\
&\qquad+y_6^3y_5^3y_2\\
&\quad+y_1^3y_3^3y_2+y_2^3y_6^3y_4+y_3^3y_2^3y_6+y_4^3y_5^3y_1+y_5^3y_1^3y_3\\
&\qquad+y_6^3y_4^3y_5\\
&\quad+3(y_1^3y_2^2y_3y_4+y_2^3y_4^2y_1y_6+y_3^3y_6^2y_2y_5+y_4^3y_1^2y_2y_5\\
&\qquad+y_5^3y_3^2y_1y_6+y_6^3y_5^2y_3y_4+y_1^3y_5^2y_2y_6+y_2^3y_3^2y_4y_5\\
&\qquad+y_3^3y_1^2y_4y_6+y_4^3y_6^2y_1y_3+y_5^3y_4^2y_2y_3+y_6^3y_2^2y_1y_5\\
&\qquad+y_1^3y_6^2y_2y_4+y_2^3y_5^2y_1y_4+y_3^3y_4^2y_5y_6+y_4^3y_3^2y_1y_2\\
&\qquad+y_5^3y_2^2y_3y_6+y_6^3y_1^2y_3y_5)\\
&\quad-2(y_1^3y_3^2y_6^2+y_2^3y_5^2y_6^2+y_3^3y_2^2y_4^2+y_4^3y_3^2y_5^2\\
&\qquad+y_5^3y_1^2y_2^2+y_6^3y_1^2y_4^2+y_1^3y_4^2y_5^2+y_2^3y_1^2y_3^2\\
&\qquad+y_3^3y_1^2y_5^2+y_4^3y_2^2y_6^2+y_5^3y_4^2y_6^2+y_6^3y_2^2y_3^2)\\
&\quad-(y_1^3y_3y_4y_5y_6+y_2^3y_1y_3y_5y_6+y_3^3y_1y_2y_4y_6+y_4^3y_1y_3y_5y_6\\
&\qquad+y_5^3y_1y_2y_4y_6+y_6^3y_1y_2y_3y_4)\\
&+y_1^2y_2^2y_6^2y_3+y_2^2y_4^2y_5^2y_6+y_3^2y_4^2y_6^2y_2\\
&\qquad+y_4^2y_3^2y_1^2y_5+y_5^2y_3^2y_2^2y_1+y_6^2y_5^2y_1^2y_4\\
&+2(y_1^2y_3^2y_2y_5y_6+y_2^2y_6^2y_3y_4y_5+y_3^2y_2^2y_1y_4y_6+y_4^2y_5^2y_1y_3y_6\\
&\qquad+y_5^2y_1^2y_2y_3y_4+y_6^2y_4^2y_1y_2y_5)\\
&-5(y_1^2y_2^2y_4y_5y_6+y_2^2y_4^2y_1y_3y_5+y_3^2y_6^2y_1y_4y_5+y_4^2y_1^2y_2y_3y_6\\
&\qquad+y_5^2y_3^2y_2y_4y_6+y_6^2y_5^2y_1y_2y_3)\}=0 \qquad (2)
\end{aligned}
$$

If this equation is isodyadic $y_1y_6=y_2y_5=y_3y_4\neq 0$ (3)

Writing $[y_1^2y_5]$ for $y_1^2y_5+y_2^2y_3+y_3^2y_1+y_4^2y_6+y_5^2y_4+y_6^2y_2$,

$[y_1y_2y_4]$ for $y_1y_2y_4 + y_3y_5y_6$ with a like notation for the other cyclosymmetric functions of $y_1, y_2, \cdots y_6$ in (2), if that equation be isodyadic, then will

$$\begin{aligned}
&x^7 - 21[y_1y_6]x^5 - 7\{[y_1^2\,y_5] + 2[y_1y_2y_4]\}x^4\\
&\quad - 7\{[y_1^3\,y_4] + 3[y_1^2\,y_2y_3] - 9y_1^2\,y_6^2\}x^3\\
&\quad - 7\{[y_1^4\,y_3] + 2[y_1^3\,y_2^2] + y_1y_6[y_1^2\,y_5] - 6y_1y_6[y_1y_2y_4]\}x^2\\
&\quad - 7\{[y_1^5\,y_2] - [y_1^4\,y_5^2] + 3[y_1^3\,y_3^2\,y_5] - 4y_1y_6[y_1^2\,y_2y_3]\\
&\qquad\qquad + [y_1y_2y_4]^2 + 7y_1^3\,y_6^3\}x\\
&\quad - [y_1^7] + 7\{-2y_1y_6[y_1^4\,y_3] + [y_1^5\,y_4y_5] + 3y_1y_6[y_1^3\,y_2^2]\\
&\qquad - [y_1^4\,y_4^2\,y_2] + [y_1^3\,y_3^3\,y_2] - 2[y_1^3\,y_4^2\,y_5^2]\\
&\qquad + 2y_1^2\,y_6^2[y_1^2] - 6y_1^2\,y_6^2[y_1y_2y_4]\} = 0 \qquad (4)
\end{aligned}$$

If now

$$\begin{aligned}
&y_1y_6 = y_2y_5 = y_3y_4 = p \neq 0\\
&y_1^2\,y_5 + y_2^2\,y_3 + y_4^2\,y_6 = p(r+\rho) \qquad y_3^2\,y_1 + y_6^2\,y_2 + y_5^2\,y_4 = p(r-\rho)\\
&y_1y_2y_4 = p(s+\sigma) \qquad y_3y_6y_5 = p(s-\sigma)\\
&y_1^4\,y_3 + y_2^4\,y_6 + y_4^4\,y_5 = p(t+\tau) \qquad y_3^4\,y_2 + y_6^4\,y_4 + y_5^4\,y_1 = p(t-\tau)\\
&y_1^7 + y_2^7 + y_3^7 + y_4^7 + y_5^7 + y_6^7 = 2pk
\end{aligned}$$

then will

$$\begin{aligned}
&[y_1^3\,y_4] = 2p(rs + \rho\sigma)\\
&[y_1^2\,y_2y_3] = 2p(rs - \rho\sigma)\\
&[y_1^3\,y_2^2] = 4p(rs - \rho\sigma) - 2p^2r\\
&[y_1^5\,y_2] = 4pst - p^2(r^2 - \rho^2 - 3p)\\
&[y_1^4\,y_5^2] = 2p^2(r^2 + \rho^2 - 2rs + 2\rho\sigma)\\
&[y_1^3\,y_3^2\,y_5] = p^2(r^2 - \rho^2 - 3p)\\
&[y_1^5\,y_4y_5] = 2p^2\{(r^2 + \rho^2)s - 2r\rho\sigma - 2pr\}\\
&[y_1^4\,y_4^2\,y_2] = 2p^2\{2(rs + \rho\sigma)s - pr\}\\
&[y_1^3\,y_3^3\,y_2] = 2p^2\{(r^2 + \rho^2)s - 2r\rho\sigma - 2pr\}\\
&[y_1^3\,y_4^2\,y_5^2] = 2p^2\{(r^2 - \rho^2 - 3p)s - t\}
\end{aligned}$$

Substituting these values in (4), it becomes

$$\begin{aligned}
&x^7 - 7p\{3x^5 + (2r + 4s)x^4 + (8rs - 4\rho\sigma - 9p)x^3\\
&\qquad + (8rs^2 - 8\rho\sigma s - 2pr - 12ps + 2t)x^2\\
&\qquad + (-4p\rho^2 - 4prs + 4p\rho\sigma + 4ps^2 + 4st + p^2)x\}\\
&\qquad - 2pk + 56p^2(\rho^2 - \rho\sigma)s = 0 \qquad (5)
\end{aligned}$$

Equating the coefficients of equal powers of x in (1) and (5) we obtain

$$\begin{aligned}
a_1 &= 2r + 4s\\
a_2 &= 8rs - 4\rho\sigma - 9p\\
a_3 &= 8rs^2 - 8\rho\sigma s - 2pr - 12ps + 2t
\end{aligned}$$

$$a_4 = 4p(-\rho^2 - rs + \rho\sigma + s^2) + 4st + p^2$$
$$a_5 = -2k + 56p(\rho^2 - \rho\sigma)s$$

To these adjoin the equations of interrelation

$$s^2 - \sigma_2 = p$$
$$t^2 - \tau^2 = p\{(r+\rho)^3(s-\sigma) + (r-\rho)^3(s+\sigma) - 6p^2(r^2-\rho^2) + 9p^3$$
$$2(st - \sigma\tau) = p(r^2 - \rho^2 - 3p)$$
$$k = p(2rs^2 + 2\rho\sigma s - pr) + (rs + \rho\sigma)t + (r\sigma + \rho s)\tau - (r^2+\rho^2)(s^2+3\sigma^2)s + 2r\rho\sigma(3s^2+\sigma^2)$$

and from these nine equations eliminate r, ρ, σ, t, τ and k. The three remaining equations arranged according to powers of s are

$$16384s^{10} - 28672a_1s^9 + (165888p + 11264a_1^2 + 6144a_2)s^8$$
$$-(89088pa_1 + 1280a_1^3 + 7168a_1a_2 - 2084a_3)s^7$$
$$+(-275712p^2 + 9600pa_1^2 + 18944pa_2 + 1408a_1^2a_2 + 768a_2^2 + 1024a_1a_3)s^6$$
$$+(222272p^2a_1 + 1152pa_1^3 + 1152pa_1a_2 - 17920pa_3 - 448a_1a_2^2 - 384a_1^2a_3)s^5$$
$$-(127456p^3 + 64768p^2a_1^2 + 46496p^2a_2 + 144pa_1^4 + 2432pa_1^2a_2 + 160pa_2^2 - 3840pa_1a_3 - 32a_2^3 - 256a_1a_2a_3 + 256a_3^2)s^4$$
$$+(104608p^3a_1 + 6848p^2a_1^3 + 26688p^2a_1a_2 + 9696p^2a_3 + 192pa_1^3a_2 + 928pa_1a_2^2 + 256pa_1^2a_3 - 1600pa_2a_3 - 32a_2^2a_3)s^3$$
$$-(123424p^4 + 17496p^3a_1^2 + 25216p^3a_2 + 96p^2a_1^4 + 4016p^2a_1^2a_2 + 2720p^2a_2^2 + 4352p^2a_1a_3 + 88pa_1^3a_2^2 + 64pa_2^3 + 256pa_1a_2a_3 - 512pa_3^2)s^2$$
$$+(40880p^4a_1 - 320p^3a_1^3 + 9328p^3a_1a_2 + 6176p^3a_3 + 64p^2a_1^3a_2 + 720p^2a_1a_2^2 + 128p^2a_1^2a_3 + 1600p^2a_2a_3 + 16pa_1a_2^3 + 32pa_2^2a_3)s$$
$$-46305p^5 + 1400p^4a_1^2 - 15876p^4a_2 - 16p^3a_1^4 + 240p^3a_1^2a_2 - 1638p^3a_2^2 - 512p^3a_1a_3 - 8p^2a_1^2a_2^2 - 68p^2a_2^3 - 256p^2a_3^2 - pa_2^4 = 0 \qquad \text{(i)}$$

$$128s^6 - 32a_1s^5 + 16(14p + a_2)s^4 - 8(14pa_1 + a_3)s^3$$
$$+4(56p^2 + 4pa_1^2 + 5pa_2 + a_4)s^2 - 8p(7pa_1 + a_1a_2 - a_3)s + p\{(7p + a_2)^2 - 4a_4\} = 0 \qquad \text{(ii)}$$

$$3328s^7 - 1664a_1s^6 - (17568p - 208a_1^2 - 512a_2)s^5 + (8104pa_1 - 128a_1a_2 - 240a_3)s^4$$
$$-(15680p^2 + 904pa_1^2 + 2172pa_2 - 12a_2^2 - 60a_1a_3 - 24a_4)s^3$$
$$+(3500p^2a_1 - 6pa_1^3 + 502pa_1a_2 + 54pa_3 - 10a_2a_3 - 6a_1a_4 + 16a_5)s^2$$
$$-(2030p^3 + 47p^2a_1^2 + 743p^2a_2 - pa_1^2a_2 + 57pa_2^2 + 36pa_1a_3 - pa_4 - a_2a_4)s$$

$$-28p^3a_1 + 2p^2a_1^3 - 2p^2a_1a_2 + 36p^2a_3 + 4pa_2a_3 + 2pa_1a_4 - 16pa_5 = 0 \quad \text{(iii)}$$

If (1) is isodyadic these three equations must have a common root. Determine it and then determine σ, r, ρ, t and τ from

$$\sigma^2 = s^2 - p$$
$$2r = -4s + a_1$$
$$4\rho\sigma = -16s^2 + 4a_1s - 9p - a_2$$
$$2t = -16s^3 + 4a_1s^2 - (10p + 2a_2)s + pa_1 + a_3$$
$$2\sigma\tau = 2st - p(r^2 - \rho^2 - 3p)$$

Let

$$\{p^3(s+\sigma)z - y_1^7\}\{p^3(s+\sigma)z - y_2^7\}\{p^3(s+\sigma)z - y_4^7\} = 0 \quad (6)$$

Expand and for $y_1^7 + y_2^7 + y_4^7$, $y_1^7y_2^7 + y_2^7y_4^7 + y_4^7y_1^7$ and $y_1^7y_2^7y_4^7$ substitute their values in terms of p, r, ρ, s, σ, t and τ, then will

$$p^2(s-\sigma)^2z^3 - (s-\sigma)^2[(r+\rho)\{p(s+\sigma) + t + \tau\} - (r-\rho)^2(s+\sigma)^2]z^2 + (s+\sigma)^2[(r-\rho)\{p(s-\sigma) + t - \tau\} - (r+\rho)^2(s-\sigma)^2]z - p^2(s+\sigma)^2 = 0 \quad (7)$$

If now

$$g_4 = p(rs + \rho\sigma) + rt + \rho\tau - (r^2+\rho^2)(s^2+\sigma^2) + 4rs\rho\sigma$$
$$\sigma g_3 = p(r\sigma + \rho s) + r\tau + \rho t - 2(r^2+\rho^2)s\sigma + 2r\rho(s^2+\sigma^2)$$

g_4 and g_3 will be rational in terms of p, a_1, a_2, a_3 and s, and (7) will be

$$p^2(s-\sigma)^2z^3 - (s-\sigma)^2(g_4 + \sigma g_3)z^2 + (s+\sigma)^2(g_4 - \sigma g_3)z - p^2(s+\sigma)^2 = 0$$

Multiplying this equation by its conjugate as to its coefficients, viz.,

$$p^2(s+\sigma)^2z^3 - (s+\sigma)^2(g_4 - \sigma g_3)z^2 + (s-\sigma)^2(g_4 + \sigma g_3)z - p^2(s-\sigma)^2 = 0$$

and transforming the product by the substitution

$$u = p^2(z + z^{-1}) \quad (8)$$

we obtain

$$u^3 - 2g_4u^2 + \{2(s^4 + 6s^2\sigma^2 + \sigma^4)g_4 - 8(s^2+\sigma^2)s\sigma^2g_3 + g_4^2 - \sigma^2g_3^2 - 3p^4\}u + 4p^4g_4 - 2(s^4 + 6s^2\sigma^2 + \sigma^4)(p^4 + g_4^2 + \sigma^2g_3^2) + 16(s^2+\sigma^2)s\sigma^2g_3g_4 = 0 \quad (9)$$

an equation with coefficients which are rational functions of p, a_1, a_2, a_3 and s.

If u_1, u_2 and u_4 are the roots of this equation taken in any order, then will

$$\begin{aligned}
y_1^7 &= \tfrac{1}{2}p(s+\sigma)\{u_1 + (u_1^2 - 4p^4)^{\frac{1}{2}}\} \\
y_2^7 &= \tfrac{1}{2}p(s+\sigma)\{u_2 + (u_2^2 - 4p^4)^{\frac{1}{2}}\} \\
y_4^7 &= \tfrac{1}{2}p(s+\sigma)\{u_4 + (u_4^2 - 4p^4)^{\frac{1}{2}}\} \\
y_6^7 &= \tfrac{1}{2}p(s+\sigma)\{u_1 - (u_1^2 - 4p^4)^{\frac{1}{2}}\} \\
y_5^7 &= \tfrac{1}{2}p(s+\sigma)\{u_2 - (u_2^2 - 4p^4)^{\frac{1}{2}}\} \\
y_3^7 &= \tfrac{1}{2}p(s-\sigma)\{u_4 - (u_4^2 - 4p^4)^{\frac{1}{2}}\}
\end{aligned} \tag{A}$$

and

$$\begin{aligned}
x_n &= \omega^n y_1 + \omega^{2n} y_2 + \omega^{3n} y_3 + \omega^{4n} y_4 + \omega^{5n} y_5 + \omega^{6n} y_6 \\
&\quad (\omega^7 - 1)/(\omega - 1) = 0 \\
&\quad n = 1, 2, 3, 4, 5, 6, 0
\end{aligned} \tag{B}$$

Changing the order of the roots of the coresolvent cubic (9) is the same as changing the ω- root used in (B), but for each order the signs of the radicals $(u^2 - 4p^4)^{\frac{1}{2}}$ in (A) must be determined so as to satisfy the equation $y_1 y_2 y_4 = p(s+\sigma)$. Changing the sign of σ (and therefore of ρ and τ) merely interchanges ω^m and ω^{7-m} in which $m = 1, 2, 3$.

Equation (i) is a quadratic in a_3, there will therefore be another number, say α_3, which substituted for a_3 will satisfy (i), a_3 and α_3 being quadratic conjugates in terms of p, a_1, a_2, and s. To determine α_3 substitute α_3 for a_3 in (i) and from this α_3-equation subtract equation (i), member from member, and divide the remainder by $(s^2 - p)(\alpha_3 - a_3)$; the resulting equation may be written

$$\begin{aligned}
8(s^2 - p)(\alpha_3 + a_3) &= 64s^5 + 32a_1 s^4 - 4(124p + 3a_1^2)s^3 \\
&\quad + 8(19p + a_2)a_1 s^2 - (193p^2 + 4pa_1^2 + 50pa_2 + a_2^2)s + 16p^2 a_1
\end{aligned} \tag{iv}$$

which expresses α_3 as a rational function of p, a_1, a_2, a_3, and s. The corresponding conjugate values of a_4 and a_5, say α_4 and α_5, may now be determined by substituting α_3, α_4 and α_5 for a_3, a_4 and a_5 in (ii) and (iii). This substitution will give

$$\alpha_4 - a_4 = 2s(\alpha_3 - a_3) \tag{v}$$

$$\begin{aligned}
4(s^2 - p)(\alpha_5 - a_5) &= \{48s^4 - 12a_1 s^3 - 2(7p - a_2)s^2 \\
&\quad + 8pa_1 s - 9p^2 - pa_2\}(\alpha_3 - a_3)
\end{aligned} \tag{vi}$$

Hence every isodyadic equation

$$x^7 - 7p(3x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x) + pa_5 = 0 \tag{1}$$

will have a conjugate

$$x^7 - 7p(3x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x) + p\alpha_5 = 0 \tag{10}$$

either of them determinable from the other by means of the equations (iv),

(v) and (vi). Equation (10) may be called the quadratic conjugate of (1). If $a_3 = a_3$, equation (1) is self-conjugate.

In like manner there may be found three quartic conjugates of a_1, say a_1', a_1'', a_1''', and three of a_2, say a_2', a_2'', a_2''' all satisfying (1), the determining equations being cubics in a_1 and a_2 respectively. However, if p, a_1, a_2, and a_3 are rational numbers these quartic conjugates will, in general, be irrational.

Hence every isodyadic septimic equation has 31 conjugates all determinable from it by the solution of equations of no degree higher than a cubic.

Examples

1. $x^7 - 7 \times 29(3x^5 - 3x^4 - 349x^3 - 124x^2 + 6508x) + 29 \times 70643 = 0$

Here $p = 29$, $a_1 = -3$, $a_2 = -349$, $a_3 = -124$, $a_4 = 6508$, $a_5 = 70643$

The diacrinic equations (i), (ii) and (iii) are

$$1024s^{10} + 5376s^9 + 172992s^8 + 1648s^7 - 20724928s^6 - 18634168s^5 + 521765923s^4 + 210298272s^3 - 4287155782s^2 + 968852648s + 4946464547 = 0$$

$$32s^6 + 24s^5 + 228s^4 + 2684s^3 + 4043s^2 - 32596s - 34191 = 0$$

$$416s^7 + 624s^6 - 85786s^5 - 101163s^4 + 1275014s^3 + 879614s^2 - 4460912s - 4050343 = 0$$

The common root of these is -1

$\therefore$ $s = -1$ $\quad$ $\sigma = 2i\sqrt{7}$

$2r = 1$ $\quad$ $2\rho = -3i\sqrt{7}$

$2t = -615$ $\quad$ $2\tau = -191i\sqrt{7}$

$2g_4 = -2045$

$4g_3 = 929$

The coresolvent cubic is

$$u^3 + 2045u^2 + 29 \times 19892u - 29^2 \times 379877 = 0$$

of which the roots are

$$u_1 = \tfrac{1}{6}\left[-4090 + 7\{\sqrt[3]{(127247 + 8733i\sqrt{3})} + \sqrt[3]{(127247 - 8733i\sqrt{3})}\}\sqrt[3]{196}\right]$$

$$u_2 = \tfrac{1}{12}\left[-8180 - 7\{(1 + i\sqrt{3})\sqrt[3]{(127247 + 8733i\sqrt{3})} + (1 - i\sqrt{3})(127247 - 8733i\sqrt{3})\}\sqrt[3]{196}\right]$$

$$u_4 = \tfrac{1}{12}\left[-8180 - 7\{(1 - i\sqrt{3})\sqrt[3]{(127247 + 8733i\sqrt{3})} + (1 + i\sqrt{3})(127247 - 8733i\sqrt{3})\}\sqrt[3]{196}\right]$$

and $\therefore$

$$2y_1^7 = 29(-1+2i\sqrt{7})\{u_1 + \sqrt{(u_1^2 - 4\times 29^4)}\}$$
$$2y_2^7 = 29(-1+2i\sqrt{7})\{u_2 + \sqrt{(u_2^2 - 4\times 29^4)}\}$$
$$2y_3^7 = 29(-1-2i\sqrt{7})\{u_4 - \sqrt{(u_4^2 - 4\times 29^4)}\}$$
$$2y_4^7 = 29(-1+2i\sqrt{7})\{u_4 + \sqrt{(u_4^2 - 4\times 29^4)}\}$$
$$2y_5^7 = 29(-1-2i\sqrt{7})\{u_2 - \sqrt{(u_2^2 - 4\times 29^4)}\}$$
$$2y_6^7 = 29(-1-2i\sqrt{7})\{u_1 - \sqrt{(u_1^2 - 4\times 29^4)}\}$$

If now $\omega = \frac{1}{2}\{\eta + \sqrt{(\eta^2 - 4)}\}$

in which $\eta = \frac{1}{6}\left[-2 + \{\sqrt[3]{(1+3i\sqrt{3})} + \sqrt[3]{(1-3i\sqrt{3})}\}\sqrt[3]{28}\right]$

then will

$$x_n = \omega^n y_1 + \omega^{2n} y_2 + \omega^{3n} y_3 + \omega^{4n} y_4 + \omega^{5n} y_5 + \omega^{6n} y_6$$
$$n = 1, 2, 3, 4, 5, 6, 0.$$

(This is the septimic of the quartisection of the cyclotomic equation $(\xi^{29}-1)/(\xi-1) = 0$.)

2. $x^7 - 7\times 29(3x^5 - 3x^4 - 349x^3 + 1248x^2 + 3764x) + 29\times 44232 = 0$

The diacrinic equations are

$$1024s^{10} + 5376s^9 + 172992s^8 + 177264s^7 - 20988352s^6$$
$$- 63493080s^5 + 491428259s^4 + 1969641312s^3$$
$$- 2581046342s^2 - 16608849144s - 12592085853 = 0$$
$$32s^6 + 24s^5 + 228s^4 - 60s^3 + 1299s^2 + 46980s + 45385 = 0$$
$$416s^7 + 624s^6 - 85786s^5 - 142323s^4 + 1235912s^3 + 1687722s^2$$
$$- 3814014s - 4209495 = 0$$

The common root of these is -1

$\therefore$

$$s = -1 \qquad \sigma = 2i\sqrt{7}$$
$$2r = 1 \qquad 2\rho = 3i\sqrt{7}$$
$$2t = 757 \qquad 2\tau = -93i\sqrt{7}$$
$$g_4 = -165$$
$$g_3 = -270$$

The coresolvent cubic is

$$u^3 + 330u^2 - 29\times 65172u + 29^2\times 1831616 = 0$$

of which the roots are

$$u_1 = -110 + \{\sqrt[3]{(72956239 + \sqrt{3484223857754977})}$$
$$+ \sqrt[3]{(72956239 - \sqrt{3484223857754977})}\}\sqrt[3]{12}$$
$$u_2 = -110 - \tfrac{1}{2}\{(1-i\sqrt{3})\sqrt[3]{(72956239 + \sqrt{3484223857754977})}$$
$$+ (1+i\sqrt{3})\sqrt[3]{(72956239 - \sqrt{3484223857754977})}\}\sqrt[3]{12}$$

$$u_4 = -110 - \tfrac{1}{2}\{(1+i\sqrt{3})\sqrt[3]{(72956239+\sqrt{3484223857754977})} + (1-i\sqrt{3})\sqrt[3]{(72956239-\sqrt{3484223857754977})}\}\sqrt[3]{12}$$

The remainder of the solution is a repetition in form of the solution of the preceding example.

(The equation of this example is the quadratic conjugate of the preceding septimic.)

3. $x^7 - 7(3x^5 + 4x^4 - 5x^3 - 4x^2 + 2x) - 5 = 0$

The diacrinic equations are

$$64s^9 - 448s^8 - 1232s^7 - 1184s^6 - 1276s^5 + 3872s^4 - 3229s^3 + 1056s^2 - 72s + 16 = 0$$
$$32s^6 - 32s^5 + 36s^4 - 104s^3 + 97s^2 - 24s - 1 = 0$$
$$832s^7 - 1664s^6 - 4200s^5 + 8984s^4 - 4974s^3 + 758s^2 - s + 32 = 0$$

The common root of these is $\frac{1}{2}$,

$\therefore$ $2s = 1$ $\quad$ $2\sigma = -i\sqrt{3}$
$r = 1$ $\quad$ $\rho = 0$
$t = 1$ $\quad$ $\tau = i\sqrt{3}$
$g_4 = 2$
$g_3 = -2$

and $y_1y_2y_4 = s + \sigma = \frac{1}{2}(1 - i\sqrt{3})$

The coresolvent cubic is

$$u^3 - 4u^2 + 5u - 2 = 0$$

$\therefore$ $u_1 = 1$, $u_2 = 1$ and $u_4 = 2$

$\therefore$ $y_1^7 = \frac{1}{4}(1-i\sqrt{3})(1+i\sqrt{3}) = 1$ $\quad$ $y_6^7 = \frac{1}{4}(1+i\sqrt{3})(1-i\sqrt{3}) = 1$
$y_2^7 = \frac{1}{4}(1-i\sqrt{3})(1+i\sqrt{3}) = 1$ $\quad$ $y_5^7 = \frac{1}{4}(1+i\sqrt{3})(1-i\sqrt{3}) = 1$
$y_4^7 = \frac{1}{2}(1-i\sqrt{3})$ $\quad$ $y_3^7 = \frac{1}{2}(1+i\sqrt{3})$

Writing ϵ for $\frac{1}{2}(-1+i\sqrt{3})$ then will $\epsilon^2 = \frac{1}{2}(-1-i\sqrt{3})$ and $\epsilon^7 = \epsilon$ and $\therefore$

$y_1 = 1$, $y_2 = 1$, $y_3 = -\epsilon^2$, $y_4 = -\epsilon$, $y_5 = 1$, $y_6 = 1$ and $y_1y_2y_4 = -\epsilon$

Hence $x_n = \omega^n + \omega^{2n} - \epsilon^2\omega^{3n} - \epsilon\omega^{4n} + \omega^{5n} + \omega^{6n}$

$n = 1, 2, 3, 4, 5, 6, 0.$

This is a reducible septimic, it may be written in the form

$$(x-5)\{x(x+3)(x^2-1)(x^2+2x-1)+1\} = 0$$

Its quadratic conjugate is

$$(x+1)(x^6-x^5-20x^4-8x^3+43x^2+27x+1)=0$$

of which the coresolvent cubic is

$$u^3+2u^2-u-2=0$$
$$\therefore \quad u_1=1,\ u_2=-1,\ u_4=-2$$

With the condition $y_1y_2y_4=\frac{1}{2}(1-i\sqrt{3})$, these give

$$y_1^7=\tfrac{1}{4}(1-i\sqrt{3})(1+i\sqrt{3})=1 \qquad y_6^7=1$$
$$y_2^7=\tfrac{1}{4}(1-i\sqrt{3})(-1-i\sqrt{3})=-1 \qquad y_5^7=-1$$
$$y_4^7=\tfrac{1}{4}(1-i\sqrt{3})(-2)=\epsilon \qquad y_3^7=\epsilon^2$$
$$\therefore \quad x_n=\omega^n-\omega^{2n}+\epsilon^2\omega^{3n}+\epsilon\omega^{4n}-\omega^{5n}+\omega^{6n}$$
$$n=1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 0.$$

Note the sign of the radical $\sqrt{(u_2^2-4p)}$ in the value of y_2^7, necessitated by the condition $y_1y_2y_4=-\epsilon$.

4. $x^7-7(3x^5+x^4-20x^3-13x^2+41x)-254=0$

The diacrinic equations are

$$16384s^{10}-28672s^9+54272s^8+26368s^7-379264s^6+259136s^5$$
$$+439568s^4-433920s^3-91032s^2+180368s-43209=0$$
$$128s^6-32s^5-96s^4-8s^3+4s^2+5=0$$
$$1664s^7-832s^6-13800s^5+6892s^4+15930s^3-7079s^2$$
$$-5174s+2366=0$$

The common root of these is $\frac{1}{2}$.

$$\therefore \quad 2s=1 \qquad 2\sigma=-i\sqrt{3}$$
$$2r=-1 \qquad 2\rho=3i\sqrt{3}$$
$$t=1 \qquad \tau=-i\sqrt{3}$$
$$2g_4=1$$
$$g_3=-1$$

and $\therefore \quad y_1y_2y_4=\frac{1}{2}(1-i\sqrt{3})=-\epsilon$

The coresolvent cubic is

$$u^3-u^2-u+1=0$$
$$\therefore \quad u_1=1 \quad u_2=1, \quad u_4=-1$$
$$\therefore \quad y_1^7=\tfrac{1}{4}(1-i\sqrt{3})(1+i\sqrt{3})=1$$
$$y_2^7=\tfrac{1}{4}(1-i\sqrt{3})(1-i\sqrt{3})=\epsilon^2$$
$$y_4^7=\tfrac{1}{4}(1-i\sqrt{3})(-1+i\sqrt{3})=-\epsilon^2$$

$$y_6^7 = \tfrac{1}{4}(1+i\sqrt{3})(1-i\sqrt{3}) = 1$$
$$y_5^7 = \tfrac{1}{4}(1+i\sqrt{3})(1+i\sqrt{3}) = \epsilon$$
$$y_3^7 = \tfrac{1}{4}(1+i\sqrt{3})(-1-i\sqrt{3}) = -\epsilon$$
$$\therefore\quad x_n = \omega^n + \epsilon^2\omega^{2n} - \epsilon\omega^{3n} - \epsilon^2\omega^{4n} + \epsilon\omega^{5n} + \omega^{6n}$$
$$n = 1, 2, 3, 4, 5, 6, 0.$$

This equation may be written in the form

$$(x-2)\{(x+1)(x+2)(x^2-9)(x^2-x-7)+1\} = 0$$

It is self-conjugate with respect to a_3.

Adyadic Septimics. If $y_1y_6 = y_2y_5 = y_4y_3 = 0$, the septimic is adyadic and cannot be reduced to the form of equation (1), the linear transformation which deprives the full-termed septimic of its second term depriving it in this case of its third term also, thus reducing it to the form

$$x^7 - 7(c_3x^4 + c_4x^3 + c_5x^2 + c_6x) - c_7 = 0 \qquad (11)$$

Hence the method adopted above for the determination and solution of isodyadic septimics is not applicable to adyadic septimics. However, if these are treated as degenerate forms of the general metacyclic septimic their determination and solution is comparatively simple as the following outlines will show.

The adyadic septimics fall into two classes and three subclasses, viz.:

The diplotriadic	in which	$y_4 = y_5 = y_6 = 0,\ y_1y_2y_3 \neq 0$
The monotriadic	" "	$y_3 = y_5 = y_6 = 0,\ y_1y_2y_4 \neq 0$
The atriadic, (Euler's)	" "	$y_3 = y_4 = y_5 = y_6 = 0,\ y_1y_2 \neq 0$
The atetradic, (Waring's)	" "	$y_2 = y_4 = y_5 = y_6 = 0,\ y_1y_3 \neq 0$
The binomial	" "	$y_2 = y_3 = y_4 = y_5 = y_6 = 0,\ y_1 \neq 0.$

If $y_4 = y_5 = y_6 = 0,\ y_1y_2y_3 \neq 0$, equation (2) becomes

$$x^7 - 7\{(y_2^2y_3 + y_3^2y_1)x^4 + (y_2^3y_1 + 3y_1^2y_2y_3)x^3 + (y_1^4y_3 + y_3^4y_2 + 2y_1^3y_2^2)x^2 + (y_1^5y_2 - y_2^4y_3^2 - y_3^4y_1^2 + 3y_1y_2^2y_3^3)x\} - \{y_1^7 + y_2^7 + y_3^7 - 7y_1y_2y_3(y_2^2 - y_1y_3)^2\} = 0 \qquad (12)$$

If (11) and (12) are identical, then must

$$c_3 = y_2^2y_3 + y_3^2y_1 \qquad c_4 = y_2^3y_1 + 3y_1^2y_2y_3$$
$$c_5 = y_1^4y_3 + y_3^4y_2 + 2y_1^3y_2^2 \qquad c_6 = y_1^5y_2 - y_2^4y_3^2 - y_3^4y_1^2 + 3y_1y_2^2y_3^3$$
$$c_7 = y_1^7 + y_2^7 + y_3^7 - 7y_1y_2y_3(y_2^2 - y_1y_3)^2$$

If $\rho = y_2^2y_3 - y_3^2y_1$, then will

$$4(\rho - 2c_3)(\rho - c_3)y_2^7 = c_4(\rho + c_3)^3 \qquad (13b)$$

$$4(\rho - 2c_3)(\rho + c_3)y_1^7 = c_4(\rho - c_3)(5\rho^2 - c_3^2 + 4c_6) \qquad (13a)$$

and
$$8c_4^2 y_3^7 = (\rho - 2c_3)^2(\rho - c_3)^2(\rho + c_3) \qquad (13c)$$

and the diacrinic equations will be

$$5\rho^6 - 25c_3\rho^5 + (29c_3^2 + 4c_6)\rho^4 + 5c_3(5c_3^2 - 4c_6)\rho^3 - (46c_3^4 - 24c_3^2 c_6 - 2c_4^3)\rho^2 - 4c_3(c_3^4 - 4c_3^2 c_6 + c_4^3)\rho + 2c_3^2(4c_3^4 - 16c_3^2 c_6 + c_4^3) = 0 \qquad (vii)$$

$$3\rho^4 + c_3\rho^3 - (15c_3^2 - 2c_6)\rho^2 + (c_3^3 + 2c_3c_6 - 2c_4c_5)\rho + 2c_3(c_3^3 - 6c_3c_6 + c_4c_5) = 0 \qquad (viii)$$

$$\rho^8 - 7c_3\rho^7 + 16c_3^2\rho^6 - 6c_3^3\rho^5 - (27c_3^4 - 68c_4^3)\rho^4 + (33c_3^5 - 12c_3c_4^3 - 8c_4^2c_7)\rho^3 + (2c_3^6 + 36c_3^2 c_4^3 + 8c_4^3 c_6 + 16c_3c_4^2 c_7)\rho^2 - (20c_3^7 + 4c_3^3 c_4^3 + 16c_3c_4^3 c_6 - 8c_3^2 c_4^2 c_7)\rho + 8c_3^8 + 2c_3^2 c_4^3 c_6 - 16c_3^3 c_4^2 c_7 = 0 \qquad (ix)$$

If these diacrinic equations have a common root, say ρ, and y_1, y_2 and y_3 have the values assigned by (13*a*) (13*b*) and (13*c*) the roots of (11) will be

$$x_n = \omega^n y_1 + \omega^{2n} y_2 + \omega^{3n} y_3 \qquad (C)$$
$$(\omega^7 - 1)/(\omega - 1) = 0$$
$$n = 1, 2, 3, 4, 5, 6, 0.$$

(The equation in which $y_2 = y_3 = y_6 = 0$, $y_1y_4y_5 \neq 0$ and that in which $y_2 = y_4 = y_6 = 0$, $y_1y_3y_5 \neq 0$ are mere variants of (12).)

II. If $y_3 = y_5 = y_6 = 0$, $y_1y_2y_4 \neq 0$, equation (2) becomes

$$x^7 - 7\{2y_1y_2y_4x^4 + (y_1^3 y_4 + y_2^3 y_1 + y_4^3 y_2)x^3 + 2(y_1^3 y_2^2 + y_2^3 y_4^2 + y_4^3 y_1^2)x^2 + (y_1^5 y_2 + y_2^5 y_4 + y_4^5 y_1 + y_1^2 y_2^2 y_4^2)x\} - y_1^7 - y_2^7 - y_4^7 + 7y_1y_2y_4(y_1^3 y_4 + y_2^3 y_1 + y_4^3 y_2) = 0 \qquad (14)$$

If (11) and (14) are identical

$$c_3 = 2y_1y_2y_4 \qquad c_4 = y_1^3 y_4 + y_2^3 y_1 + y_4^3 y_2$$
$$c_5 = 2(y_1^3 y_2^2 + y_2^3 y_4^2 + y_4^3 y_1^2) \qquad c_6 = y_1^5 y_2 + y_2^5 y_4 + y_4^5 y_1 + y_1^2 y_2^2 y_4^2$$
$$c_7 = y_1^7 + y_2^7 + y_4^7 - 7y_1y_2y_4(y_1^3 y_4 + y_2^3 y_1 + y_4^3 y_2)$$

and the diacrinic equations will be

$$16c_3c_6^2 - 16c_4c_5c_6 + 4c_3^3 c_6 + 4c_5^3 - 20c_3^2 c_4c_5 + 16c_3c_4^3 + 7c_3^5 = 0 \qquad (x)$$
$$2c_3c_7 - 4c_4c_6 + c_5^2 + 7c_3^2 c_4 = 0 \qquad (xi)$$

and y_1^7, y_2^7, y_4^7 will be the roots of

$$128z^3 - 64(2c_7 + 7c_3c_4)z^2 - 8(4c_3c_5c_6 - 4c_4c_5^2 + c_3^3 c_5 + 4c_3^2 c_4^2)z - c_3^7 = 0 \quad \text{(D)}$$

III. If $y_3 = y_4 = y_5 = y_6 = 0$, $y_1y_2 \neq 0$, equation (2) becomes

$$x^7 - 7y_1y_2^3 x^3 - 14y_1^3 y_2^2 x^2 - 7y_1^5 y_2 x - y_1^7 - y_2^7 = 0 \quad (15)$$

If (11) and (15) are identical

$$c_3 = 0,\ c_4 = y_1y_2^3,\ c_5 = 2y_1^3 y_2^2,\ c_6 = y_1^5 y_2,\ c_7 = y_1^7 + y_2^7$$

and $\therefore$ the diacrinic equations will be

$$c_3 = 0,\quad 4c_4c_6 - c_5^2 = 0,\quad c_5c_7 - 2(c_6^2 + c_4^3) = 0 \quad \text{(xii)}$$

and the coresolvent will be

$$c_5z^2 - (16c_6^2 + c_4^3)z + 4c_4^2 c_5c_6 = 0 \quad \text{(E)}$$

(This is Euler's Septimic; see his memoir *De resolutione æquat.* in *N. Comm. Acad. imp. sc. Petropol.*, T. ix, 1762. See also Waring's *Meditationes Algebraicæ*, 3rd edition, p. 170, § 3. 3.)

IV. If $y_2 = y_4 = y_5 = y_6 = 0$, $y_1y_3 \neq 0$, equation (2) becomes

$$x^7 - 7y_1y_3^2 x^4 - 7y_1^4 y_3x^2 + 7y_1^2 y_3^4 x - y_1^7 - y_3^7 = 0 \quad (16)$$

If (11) and (16) are identical

$$c_3 = y_1y_3^2,\ c_4 = 0,\ c_5 = y_1^4 y_3,\ c_6 = -y_1^2 y_3^4,\ c_7 = y_1^7 + y_3^7$$

and the diacrinic equations will be

$$c_4 = 0,\quad c_6 + c_3^2 = 0,\quad c_3c_5c_7 - c_5^3 - c_3^5 = 0 \quad \text{(xiii)}$$

and the coresolvent will be

$$z^2 - c_7z + c_3^3c_5 = 0 \quad \text{(E)}$$

(This is Waring's Septimic. See *Med. Algeb.*, 3rd ed., p. 170, § 4. 1.)

V. If $y_2 = y_3 = y_4 = y_5 = y_6$, $y_1 \neq 0$, equation (2) will be the binomial septimic.

If $y_2 = y_3 = y_4 = y_5 = 0$, $y_1y_6 \neq 0$, equation (2) becomes

$$x^7 - 7y_1y_6x^5 + 14y_1^2 y_2^2 x^3 - 7y_1^3 y_6^3 x - y_1^7 - y_6^7 = 0 \quad (17)$$

If this equation is identical with

$$x^7 - 7(c_2^5 x^5 + c_3x^4 + c_4x^3 + c_5x^2 + c_6x) - c_7 = 0$$

then will

$$c_2 = y_1 y_6,\ c_3 = 0,\ c_4 = -2{y_1}^2 {y_6}^2,\ c_5 = 0,\ c_6 = {y_1}^3 {y_6}^3,\ c_7 = {y_1}^7 + {y_6}^7$$

The diacrinic equations will be

$$c_3 = c_5 = 0,\ c_4 + 2{c_2}^2 = 0,\ c_6 - {c_2}^3 = 0$$

and the coresolvent will be

$$z^2 - c_7 z + {c_2}^7 = 0$$

(This is the well-known De Moivrian monodyadic septimic. See Waring's *Med. Algeb.*, 3rd ed., p. 172, § 6. 1.)

ADDENDA.

A. On changing the signs of the coefficients of the even powers of x (including x^0) in the left hand member of equation (2) it becomes the circulant of the seventh order arranged according to powers of x.

B. The equations (3) which assign the conditions necessary and sufficient for (2) to be isodyadic require that for (1), if metacyclic to be also isodyadic, there must be two independent relations among its coefficients and these two relations are given by the resultants of the elimination of s between (i) and (ii) and (i) and (iii).

C. The coresolvent equations of a metacyclic septimic are a quadratic and a cubic, (*Am. J. M.*, XXIII, 56, Th. II.) Equation (8) is the quadratic coresolvent of the solution here given.

D. Other coresolvent cubics than (9), or rather than (7) may be found; *e. g.*, if z_1, z_2 and z_4 are the roots of

$$(pz - {y_1}^2 y_5)(pz - {y_2}^2 y_3)(pz - {y_4}^2 y_6) = 0$$

which on expansion becomes

$$z^3 - (r+\rho)z^2 + (r-\rho)(s+\sigma)z - p(s+\sigma) = 0$$

and if z_6, z_5, z_3 are the roots of its conjugate

$$z^3 - (r-\rho)z^2 + (r+\rho)(s-\sigma)z - p(s-\sigma) = 0$$

we may assume

$$pz_1 = {y_1}^2 y_5 \qquad pz_2 = {y_2}^2 y_3 \qquad pz_4 = {y_4}^2 y_6$$
$$pz_6 = {y_6}^2 y_2 \qquad pz_5 = {y_5}^2 y_4 \qquad pz_3 = {y_3}^2 y_1$$

then will

$${y_1}^7 = p(s+\sigma){z_1}^3 z_2 \qquad {y_6}^7 = p(s-\sigma){z_6}^3 z_5$$

$$y_2^7 = p(s+\sigma)z_2^3 z_4 \qquad y_5^7 = p(s-\sigma)z_5^3 z_3$$
$$y_4^7 = p(s+\sigma)z_4^3 z_1 \qquad y_3^7 = p(s-\sigma)z_3^3 z_6$$

This solution is simple in appearance but it is objectionable because it associates the roots of the given septimic and those of its conjugate, and they have to be separated by trial in the given equation. For example, the bicubic

$$(2z^3 - z^2 - 43z + 58)^2 + 7(3z^2 - z - 116)^2 = 0$$

is a coresolvent of the septimics of both Ex. 1 and Ex. 2.

E. If (4) is a Gaussian cyclotomic or one of its conjugates, $\sigma^2 + 7m^2 = 0$, hence

If p is a prime $\equiv 1 \bmod 14$, $p = u^2 + 7v^2$ is solvable in integers.

This theorem may be generalized (*Am. J. M.*, XXI, 277 and XXIII, 56);—

If n and p are primes, $n \equiv 3 \bmod 4$, $p \equiv 1 \bmod 2n$, $p^{\frac{1}{2}(n-5)} = u^2 + nv^2$ is solvable in integers. Hence, (Legendre, *Th. d. Nos.*, Sec. Partie § X),

(**a**) If n and p are primes, $n \equiv 3 \bmod 8$, $p \equiv 1 \bmod 2n$, $4p = u^2 + nv^2$ is solvable in integers.

(**b**) If n and p are primes $n \equiv 7 \bmod 8$, $p \equiv 1 \bmod 2n$, the simultaneous equations $g_1 g_2 - h^2 = n$, $gp = u^2 - nv^2$, (g either g_1 or g_2) will be solvable in integers. (Legendre, *Th. d. Nos.*, Table V, gives the values of g for all values of n up to 103. A few misprints occur in the Table and there are several omissions.)